New examples of mixing local rank one transformations

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Dynamical Systems Meeting, Lower Silesia 2010

Outline

Common settings

Automorpisms

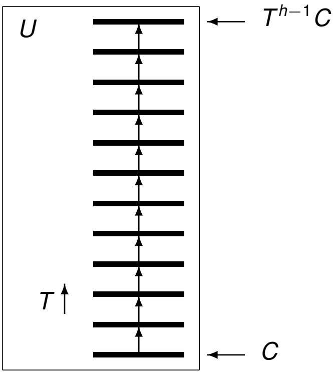
An invertible measure preserving transformation T of the standard Lebesgue space $(X, \mathcal{A}, \mu)$ is called an *automorphism* of the space X.

An automorphism generates an action of the group $\mathbb{Z}$.

Flows

A flow T^t is an action of the group $\mathbb{R}$, i.e. a family of automorphisms $T^t\colon\ X \to X$ satisfying the group property $T^{t+u} = T^t T^u$.

Tower definition of rank one transformation



We associate to a *Rokhlin tower* partition

$$\xi = \{C, TC, T^2C, \ldots, T^{h-1}C, E\},$$

$$E = X \setminus U.$$

T is said to be a *rank one* map if there exist a sequence of Rokhlin towers U_n such that corresponding partitions ξ_n asymptotically approximate σ-algebra of the phase space.

Tower definition of rank one transformation

Definition. T is called a *rank one* transformation if there exist a sequence of Rokhlin towers U_n identified with the corresponding *tower partitions*

$$\xi_n = \{B_n, TB_n, T^2B_n, \ldots, T^{h_n-1}B_n, E_n\}$$

such that $\mu(U_n) \to 1$ and for any measurabe set A there exist ξ_n-measurable sets A_n such that $\mu(A \triangle A_n) \to 0$.

Symbolic representation

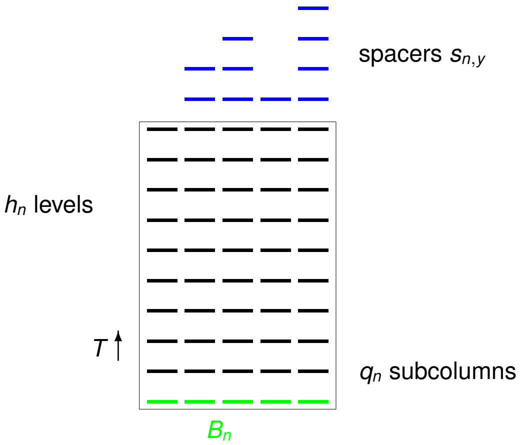
Lemma. We can assume that the *n*-th tower in the definition of a rank one map refines the previous tower.

Definition. Consider a sequence of words W_n such that

$$W_{n+1} = W_n 1^{s_{n,1}} W_n 1^{s_{n,2}} W_n \dots 1^{s_{n,q_n}} W_n,$$

where symbol 1 is used to create *spacers* between words.
It defines a rank one transformation.

Cutting-and-stacking construction



$$\prod_{n=1}^{\infty} \frac{h_{n+1}}{q_n h_n} < \infty$$

Cutting-and-stacking construction: A formal scheme

Let $X_n = [0, h_n] \cap \mathbb{Z}$, and for spacers $s_{n,y}$ set

$$\omega_n(y+1) = \omega_n(y) + h_n + s_{n,y}, \qquad 0 \le y < q_n.$$

Define projections $\phi_n\colon\ X_{n+1} \to X_n$ as follows

- $\phi_n(\omega_n(y) + x') = x'$ if $x' \in [0, h_n)$,
- $\phi_n(t) = h_n$ elsewise.

Let X be the inverse limit of X_n:

$$X = \{x = (x_1, x_2, \dots, x_n, \dots)\colon\ x_{n+1} = \phi_n(x_n),\ x_n \in X_n\}$$

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Spectral invariants of an automorphism

The unitary Koopman operator in $L^2(X, \mu)$ associated with a transformation T is

$$\widehat{T}\colon\ L^2(X,\mu) \to L^2(X,\mu)\colon\ f(x) \to f(Tx)$$

The spectral invariants of T are the

- maximal spectral type σ on S^1 and the
- multiplicity function $\mathcal{M}(z)\colon\ S^1 \to \mathbb{N} \sqcup \{\infty\}$.

Usually we study $\widehat{T}$ on the space of functions with zero mean.

Rank one systems: Spectral multiplicity

Theorem. Rank one transformations and rank one flows are ergodic and have spectral multiplicity 1, i.e. $\mathcal{M}(z) \equiv 1$ (in other words, *simple spectrum*).

A transformation T has simple spectrum iff there exists an element $f \in L^2(X, \mu)$ (*cyclic vector*) such that

$$L^2(X, \mu) = \overline{\mathrm{Span}}(\{\widehat{T}^k f \colon \ k \in \mathbb{Z}\}).$$

In this case $\sigma_f \sim \sigma$, where σ_f is defined by the property

$$\int_{S^1} z^k \, d\sigma_f = \left\langle T^k f, \, f \right\rangle.$$

Rank one systems: Spectral type

Question. Is the following true: Any rank one transformation has the spectral type which is singular with respect to the Lebesgue measure λ on S^1?

$$\sigma_f \perp \lambda$$

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Polynomials with Littlewood type coefficient constraints

$$\mathcal{K}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k : \ |a_k| \equiv 1 \right\}.$$

$$\mathcal{L}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k : \ a_k \in \{-1, \ 1\} \right\}.$$

$$\mathcal{M}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n}} (z^{\omega_1} + z^{\omega_2} + \ldots + z^{\omega_n}) : \ \omega_j \in \mathbb{Z}, \ \omega_j < \omega_{j+1} \right\}.$$

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Flatness phenomenon

Question (Littlewood, 1966). Is the following true?
For any $\varepsilon > 0$ there exists a polynomial with unimodular coefficients $P(z) \in \mathcal{K}_n$ such that

$$\forall z \in S^1 \qquad \big||P(z)| - 1\big| < \varepsilon.$$

Theorem (Kahane, 1980). The answer is "yes" with the speed of convergence

$$\varepsilon_n = O(n^{-1/17}\sqrt{\ln n}).$$

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Generalized Riesz products

If a function $f \in L^2(X,\mu)$ is constant on the levels of n-th tower then we identify f with a function $f_{(n)}: \mathbb{Z} \to \mathbb{C}$.
Define polynomials

$$P_n(z) = \frac{1}{\sqrt{q_n}} \sum_{y=0}^{q_n-1} z^{\omega_n(y)} \in \mathcal{M}_n.$$

The spectral measure σ_f is given (up to a constant multiplier) by the infinite product

$$\sigma_f = |\widehat{f}_{(n_0)}|^2 \prod_{n=n_0}^{\infty} |P_n(z)|^2.$$

Applications to rank one systems

Theorem (Bourgain, 1993). Ornstein rank one transformations have singular spectral type.

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Question. Does there exist an automorphism with *finite spectral multiplicity* and *absolutely continuous spectral type*?

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Rotated words system. Away from rank one

Let ρ_a be the operator that rotates a word by a positions:

$$W = W_1 W_2 \overset{\rho_a}{\mapsto} W_2 W_1 \qquad \text{if} \qquad |W_1| = a.$$

For example, $\rho_1(CAT) = ATC$, $\rho_2(CAT) = TCA$.

Observation: ρ_a is a discrete IET.

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Rotated words system

For a given word

$$W_n = abc\ldots z$$

consider all its' rotations:

$$\begin{aligned}
\rho_0(W_n) &= abc\ldots yz\\
\rho_1(W_n) &= bcd\ldots za\\
\rho_2(W_n) &= cde\ldots ab\\
&\ldots\\
\rho_{h_n-1}(W_n) &= zab\ldots xy.
\end{aligned}$$

Rotated words system

Definition. Fix a sequence q_n and a sequence $(a_{n,y})$, where $a_{n,y} \in \{0, 1, \ldots, h_n - 1\}$, $y = 0, 1, \ldots, q_n$.

Let us define a sequence of words W_n which is constructed iteratively:

$$W_{n+1} = \rho_{a_{n,0}}(W_n)\rho_{a_{n,2}}(W_n) \ldots \rho_{a_{n,q_n-1}}(W_n).$$

Common examples

Morse sequences. The sequence 0110100110010110 . . . is a rotated words system given by $h_n = 2^n$, $q_n = 2$ and $\rho_{n,0} = 0,\ \rho_{n,1} = \frac{1}{2}h_n$.

Three letter Morse sequences. $abc \mapsto abc.bca.cab$

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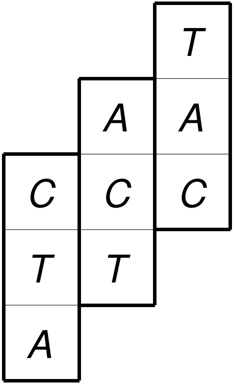
Illustration to the dynamic: $CAT \mapsto CAT.ATC.TCA.TCA.CAT.ATC$



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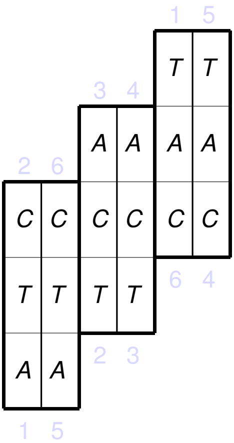

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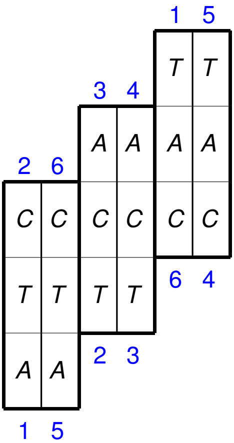

Addic representation

Given q_n set $h_{n+1} = q_n h_n$ and $X_n = \{0, \ldots, h_{n+1} - 1\}$, and consider maps $\phi_n \colon X_{n+1} \to X_n$:

$$\phi_n(yh_n + x') = \rho_{a_{n,y}}(x'), \qquad 0 \leq x' < h_n.$$

Define X to be the inverse limit of:

$$X_1 \overset{\phi_1}{\leftarrow} X_2 \overset{\phi_2}{\leftarrow} X_3 \overset{\phi_3}{\leftarrow} \ldots,$$

$$X = \{x = (x_1, x_2, \ldots) \colon \; x_n \in X_n, \; \phi_n(x_{n+1}) = x_n\}$$

Addic representation

Define $Tx = (\ldots, x_n + 1, x_{n+1} + 1, \ldots)$.

Almost surely $x_n + 1$ is correct starting from some n_0.
Let other x_n be defined to meet the rule $\phi_n(x_{n+1}) = x_n$.

Lemma. T is a measure preserving transformation.

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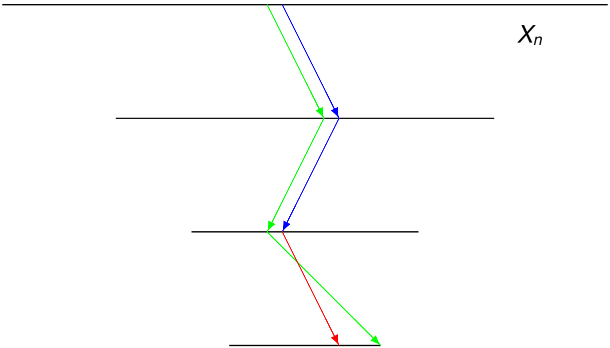
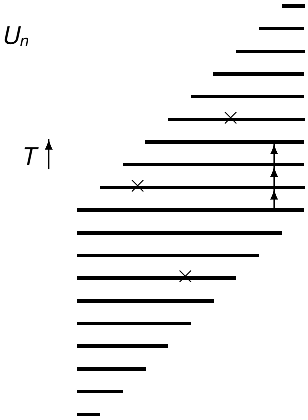
Bratteli–Vershik diagram



T maps a path $x = (x_1, x_2, \ldots)$ to the next path $(\ldots, x_n + 1, \ldots)$.

Outline

Iceberg (by a plot)



Vertical fibers correspond to different rotations of the word.

Point at the top of the iceberg is mapped into a set in the bottom.

A small ε_n-fraction of points do not satisfy lifting rule.

Iceberg partition ξ_{n+1} into levels refines ξ_n.

Spectral properties of iceberg maps

Theorem. Let T be an iceberg transformation given by uniforme i.i.d. randomized rotations $a_{n,k}$. There exist a sequence q_n such that the following properties hold a.s.

(i) T has $1/4$-local rank

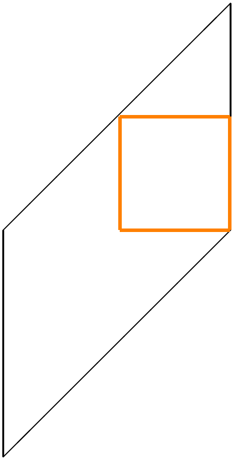
(ii) T has simple spectrum

(iii) $\sigma * \sigma \ll \lambda$

(iv) If f is a ξ_n-measurable function, $\int f\, d\mu = 0$, then $\forall \varepsilon > 0$

$$\langle T^t f, f\rangle = O(t^{-1/2+\varepsilon})$$

$\frac{1}{4}$ Local rank one



We can fit a rectangle with the maximal area $\frac{1}{4}\mu(X)$ (asymptotically).

Estimating spectral multiplicity

Lemma (*Katok, Stepin*). Let U be a unitary operator on a separable Hilbert space H, and let σ and $N(z)$ are the spectral measure and the multiplicity function of U.
If $N(x) \geq m$ on a set of positive σ-measure then there exist m orthogonal vectors $f_1, \ldots, f_m$ such that for any cyclic space $Z \subseteq H$ and any $g_1, \ldots, g_m \in Z$, $\|g_i\| \equiv a$, the following is true

$$\sum_{i=1}^{m} \|f_i - g_i\|^2 \geq m(1 + a^2 - 2a/\sqrt{m})$$

Questions

- Does iceberg have rank one or not? Finite rank?
- Does iceberg have MSJ?
- Is it true that iceberg is mixing of all orders?
- Let σ be the spectral type of the iceberg map. Is it true that σ has an absolutely continuous component?
- Is it true that σ is Lebesgue?

Outline

- Motivation
  - Rank one maps and flows
  - Spectral invariants
- Motivation II
  - Littlewood polynomials
  - Generalized Riesz products
- Iceberg transformation
  - Rotated words system
  - Iceberg map
- Rank one flow with Lebesgue spectrum
  - Exponential staircase flows
  - Discussion

Flat trigonometric sums with coefficients in $\{0, 1\}$

$$\mathcal{M}_q^{\mathbb{R}} = \left\{ \mathcal{P}(t) = \tfrac{1}{\sqrt{q}} \sum_{y=0}^{q-1} \exp(2\pi i\, t\omega(y) \colon\ \omega(y) \in \mathbb{R} \right\}.$$

Theorem. For any $0 < a < b$ and $\delta > 0$ there exists a sum $\mathcal{P}(t) \in \mathcal{M}_q$ which is δ-flat in $L^1(a, b)$ (and $L^2(a, b)$), i.e.

$$\left\| \left| \mathcal{P}(t) \right|_{(a,b)} \right| - 1 \Big\|_1 < \delta.$$

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Flat trigonometric sums with coefficients in $\{0, 1\}$

$$\omega(y) = \frac{q}{\varepsilon^2} e^{\varepsilon y / q}$$

$$\varepsilon^{-1} \in \mathbb{N}$$

The concept: $|\mathcal{P}(t)| \approx 1$ with λ-probabiity close to 1 if

$$t \to \infty, \quad \varepsilon \to \infty, \quad q = q_j \to \infty.$$

Remark. Whenever we fix an interval (t_0, t_1) for t, the degree q goes to infinity along a rare subsequence q_j.

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Exponential staircase flow

We construct a rank one flow with the following parameters:

- q_n is the number of subcolumns
- spacers $s_{n,y} = \omega_n(y+1) - \omega_n(y) - h_n$
- $\omega_n(y) = \mu_n \dfrac{q_n}{\varepsilon_n^2} e^{\varepsilon_n y / q_n}, \qquad h_n = \dfrac{\mu_n}{\varepsilon_n}$

$\mu_n \to \infty$ (*slowest*), $\varepsilon_n \to 0$, $q_n \to \infty$ (*fastest*).

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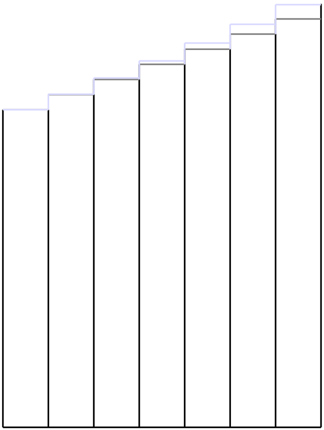

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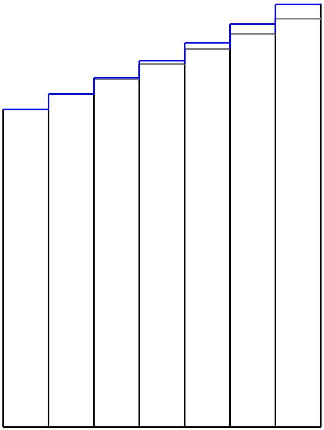

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Outline

Iceberg for $\mathbb{Z}^2$-actions

Observation. Let T and S are the generators of $\mathbb{Z}^2$-iceberg action (T^iS^j).

- Both T and S have countable Lebesgue spectrum and zero entropy
- The spectral type σ of the action satisfies: $\sigma * \sigma \ll \lambda_2$ and $\pi_X\sigma \ll \lambda$, $\pi_Y\sigma \ll \lambda$.

Iceberg approximation

Definition. T is said to have m-fold iceberg rank is the σ-algebra is approximated by a sequence of iceberge tuple $(\mathcal{I}_n^{(1)}, \ldots, \mathcal{I}_n^{(m)})$.

Group actions

Iceberg

- Iceberg is extended to general group actions (in the same cases as rank one: $\mathbb{Z}^d$, $\mathbb{R}^d$, nilpotent group actions, (C, F)-construction)
- Rotated words approach can be combined with rank one
- In place of rotation we can consider any IET

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- At this point, the effect is observed for dimension one
- Staircase flows can be extented directly to p-adic field actions
- There exist rank one $\mathbb{R}^d$-actions with Lebesgue spectral type

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