On Littlewood-type polynomials and exponential sums and applications to spectral theory

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Rouen, 08.11.2010

Polynomials with Littlewood-type coefficient constraints

Definition. A complex polynomial

$$P(z) = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$$

is called *unimodular* if $|a_k| \equiv 1$.

Definition. A polynomial $P(z)$ is called ε-*ultraflat* if

$$\forall z \in S^1 \qquad \big||P(z)| - 1\big| < \varepsilon.$$

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$$\mathcal{G}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k : \ |a_k| \equiv 1 \right\}.$$

$$\mathcal{L}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k : \ a_k \in \{-1, 1\} \right\} \subset \mathcal{G}_n.$$

$$\mathcal{M}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n}} (z^{\omega_1} + z^{\omega_2} + \ldots + z^{\omega_n}) : \ \omega_j \in \mathbb{Z}, \ \omega_j < \omega_{j+1} \right\}.$$

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Littlewood's flatness problem

Question (Littlewood, 1966). Is the following true?
For any $\varepsilon > 0$ there exists an ε-ultraflat polynomial $P(z) \in \mathcal{G}_n$, $n \geq 1$.

Theorem (Kahane, 1980). The answer is "yes" with the speed of convergence

$$\varepsilon_n = O(n^{-1/17}\sqrt{\ln n}).$$

Question (*open*).
Is it possible to find an ultraflat polynomial in $\mathcal{L}_n$?
Is it possible to find an L^p-flat polynomial in $\mathcal{M}_n$?

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Estimating Littlewood-type polynomials

Littlewood's famous conjecture: For any $f \in \mathcal{M}_n$

$$\|\sqrt{n} \cdot f\|_1 \geq c \log n.$$

This was proved by Konyagin (1980).

Littlewood noticed (1968): "*Although it is known that*

$$g_n(\theta) = \sum_{m=0}^{n} e^{im \log m} e^{im\theta}$$

satisfies $|g_n(\theta) < c\sqrt{n+1}|$ *on* $\mathbb{R}$*, the existence of polynomials* $P_n \in \mathcal{L}_n$ *with* $|P_n| \leq c$ *is shown recenty*".

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Rudin–Shapiro polynomials

The Rudin–Shapiro sequence of polynomials $P_n, Q_n \in \mathcal{L}_{\bar{\mu}_n}$ is defined reccurently as follows

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\begin{aligned}
P_0(z) &= Q_0(z) = 1 \\
P_{n+1}(z) &= P_n(z) + z^{2^n} Q_n(z), \\
Q_{n+1}(z) &= P_n(z) - z^{2^n} Q_n(z), \\
|P_n(z)|^2 + |Q_n(z)|^2 &= 2(\bar{\mu}_n + 1),
\end{aligned}
$$

where

$$\bar{\mu}_n = \deg P_n = \deg Q_n = 2^n - 1.$$

Theorem. There exist polynomials in $\mathcal{L}_{\bar{\mu}_n}$ such that $|P_n(z)| \leq c$.

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Erdös conjecture

And the following Littlewood's question still has no answer.

Question (*open*).
Is it possible to find polynomials in $\mathcal{L}_n$ modulus also at least c? $c_1 \leq |P_n(z)| \leq c_2$.

Erdös conjectured that there exists an absolute constant $c^* > 1$ such that $\max |P(z)| \geq c^*$ for all $P \in \mathcal{L}_n$.

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Behavior in average

Theorem (Salem–Zigmund, 1954).
For all but $o(\#\mathcal{L}_n)$ polynomials P_n from $\mathcal{L}_n$

$$c_1 \log n \leq \max_{|z|=1} |P_n(z)| \leq c_2 \log n.$$

Theorem (Newmann–Byrnes, 1990). The expected L^4-norm

$$\mathrm{E}\|\sqrt{n} \cdot P_n\|_4 = (2n^2 - n)^{1/4}$$

for polynomials $P_n \in \mathcal{L}_n$.

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Number of zeros at 1

Theorem (Amoroso, Bombieri–Vaaler, Hua, Erdélyi).
There is an absolute $c > 0$ such that every polynomial

$$P(z) = \sum_{j=0}^{n} a_j z^j, \quad |a_j| \le 1, \quad a_j \in \mathbb{C},$$

has at most

$$c(n(1 - \log|a_0|))^{1/2}$$

zeros at 1.

Bernstein inequalities

We say that a polynomial $P(z)$ is ε-flat in $L^p(S^1)$ if

$$\big\| |P(z)| - 1 \big\|_p < \varepsilon.$$

Theorem (Queffelec, Saffari, Nazarov).
For any $p \in (0, +\infty]$, $p \neq 2$,

$$\lim_{n\to\infty} \beta_p(\mathcal{G}_n) = \lim_{n\to\infty} \beta_p(\mathcal{L}_n) \to 1,$$

where

$$\beta_p(\mathcal{C}) = \sup_{P\in\mathcal{C}} \frac{1}{\deg P} \cdot \frac{\|P'\|_p}{\|P\|_p}$$

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Bernstein inequalities and phase behavior

Theorem (Queffelec, Saffari).
For $\varepsilon_n = K_1 n^{-1/96}$ there exist an ε_n-ultraflat $P_n \in \mathcal{G}_n$ such that

$$\frac{\|P'\|_p}{n\|P\|_p} = \gamma_p + O_\eta(\varepsilon_n),$$

where $0 < \eta \leq p \leq \infty$, and P_n can be choosen so that

$$P_n(e^{it}) = |P_n(e^{it})|\, e^{i\alpha_n(t)},$$

$$\frac{\alpha_n'(t)}{n} = \frac{t}{2\pi} + \frac{1}{2} + O_t(n^{-1/96}).$$

Uniform distribution of the angular speed

The following theorem was conjectured by Saffari (1992).

Theorem (Erdélyi, $\sim$ 2005). Let (P_n) be an ultraflat sequence of polynomials in $\mathcal{G}_n$. Then the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$,

$$\lambda\{t \in [0, 2\pi]\colon\ 0 \le \alpha_n'(t) \le nx\} = 2\pi x + \epsilon_n(x),$$

where $\epsilon_n(x) \to 0$ uniformely on $[0, 1]$.

Uniform distribution of the angular speed

Theorem: *reformulaton* (Erdélyi). Let P_n be an ultraflat sequence of unimodular polynomials, then

$$\frac{1}{2\pi}\int_0^{2\pi} |\alpha_n'(t)|^q \, dt = \frac{n^q}{q+1} + o_{n,q} n^q.$$

Theorem. At the same time, the higher derivatives $\alpha_n^{(r)}$ are small,

$$\max_{0 \le t \le 2\pi} |\alpha_n^{(r)}| \le \epsilon_{n,r} n^r, \quad r \ge 2,$$

where $\epsilon_{n,r} \to 0$ as $n \to \infty$.

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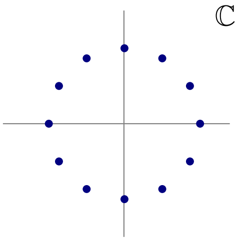
Flatness on a lattice

Example: Let $e(t) = e^{2\pi i t}$ and define polynomials

$$P^{(m)}(z) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e\left(\frac{mj^2}{n}\right) z^j,$$

where $(m, n) = 1$ and n is prime. Then

$$|P^{(m)}(z)| = 1 \quad \text{if} \quad z^n = 1.$$



Connection between $\mathcal{G}_n$ and $\mathcal{M}_n$

Consider a polynomial

$$P(z) = \sum_{j=0}^{n-1} z^{hj+s_j} \in \mathcal{M}_n, \qquad s_j \ll h,$$

and for $z = e^{2\pi i\, t}$ write

$$z^{hj+s_j} = z^{s_j} z^{hj} = a_j^{(t)} w^j,$$

where

$$w = z^h = e^{2\pi i\, ht}, \qquad a_j^{(t)} = e^{2\pi i\, t \cdot s_j}.$$

Dynamical Littlewood's flatness problem

We have constructed a family of polynomials $Q^{(t)} \in \mathcal{G}_{n-1}$,

$$Q^{(t)}(w) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} a_j^{(t)} w^j.$$

Question. Is it possible to find a family of polynomials $Q^{(t)} \in \mathcal{G}_{n-1}$ *simultaneously flat* accoring to L^p-norm, or ultraflat?

Observation. Flatness of $Q^{(t)}$ implies flatness of the underlying $P \in \mathcal{M}_n$.

Question. Is it possible to distinguish flatness of $P \in \mathcal{M}_n$ and the corresponding $Q^{(t)} \in \mathcal{G}_{n-1}$?

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Dynamical Littlewood's flatness problem

Simple case: Is it possible to find simultaneously flat (or ultraflat?) polynomials:

$$Q(z) = \sum_{j=0}^{n} a_j z^j, \quad \text{and} \quad Q^{(2)}(z) = \sum_{j=0}^{n} a_j^2 z^j \ ?$$

Question. Is it possible to ensure ε-flatness for all but ε-part of polynomials in the sequence

$$Q(z),\ Q^{(2)}(z),\ \ldots,\ Q^{(h)}(z)\ ?$$

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Spectral invariants of dynamical systems

Let T be an invertible measure preserving transformation of the standard Lebesgue space $(X, \mathcal{A}, \mu)$, $X = [0, 1]$.
The Koopman operator

$$\widehat{T}\colon\ L^2(X,\mu) \to L^2(X,\mu)\colon\ f(x) \mapsto f(Tx)$$

Spectral invariants of T are the

- maximal spectral type σ on $S^1 = \{z \in \mathbb{C}\colon\ |z| = 1\}$ and the
- multiplicity function $\mathcal{M}(z)\colon\ S^1 \to \mathbb{N} \sqcup \{\infty\}$.

Usually we study $\widehat{T}$ on the space of functions with zero mean.

Spectral invariants of dynamical systems

Examples:

- Bernoulli maps: $\sigma = \lambda$ and the multiplicity $= \infty$
- Transformation with pure point spectrum: spectrum is simple, and σ is a distribution on a discrete subgroup in S^1 (example: irrational rotation)

Problem (Banach). Is the following true?
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Banach and Kirillov problems

Banach question: *reformulation*
(closer to the original version).

Is the following true? There exists a measure preserving transformation T and an element $\xi \in L^2(X,\mu)$ such that $\widehat{T}^j\xi \perp \widehat{T}^k\xi$ and $\{\widehat{T}^j\xi\}$ generate the entire $L^2(X,\mu)$.

Question (Kirillov, 1967). Given an Abelian group G is it possible to find a G-action with simple Lebesgue spectrum?

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Finite spectral multiplicity

Theorem (Guenais, 1998). Connection between Littlewood-type problem in $\mathcal{L}_n$ (coefficients ± 1) and the spectral properties of Morse cocycles.

Theorem (Downarowicz, Lacroix, 1998).
If all continuous binary Morse systems have singular spectra then the merit factors of binary words are bounded (the Turyn's conjecture holds).
Given a word A in alphabet $\{-1, +1\}$, the merit factor of A

$$M_A = \frac{1}{\|P_A\|_4^4 - 1}, \qquad P_A(z) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} A(j) z^j.$$

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Rank one transformations

Definition. T is called a *rank one* transformation if there exist a sequence of partitions

$$\xi_n = \{B_n, TB_n, T^2B_n, \ldots, T^{h_n-1}B_n, E_n\},$$

identified with Rokhlin towers, such that $\mu(\bigcup_{j=0}^{h_n-1} T^j B_n) \to 1$ and for any measurabe set A there exist ξ_n-measurable sets A_n with $\mu(A \triangle A_n) \to 0$.

Rank one transformations: Symbolic definition

Any rank one transformation can be described in the following way using the language of symbolic dynamics.

Starting from a word W_{n_0} consider the sequence of words W_n given by

$$W_{n+1} = W_n 1^{s_{n,0}} W_n 1^{s_{n,1}} W_n 1^{s_{n,2}} \dots W_n 1^{s_{n,q_n-1}},$$

where symbol “1” is used to create *spacers* between words, and parameters $s_{n,1}$ are fixed in advance.

Generalized Riesz products

Let us define polynomials

$$P_n(z) = \frac{1}{\sqrt{q_n}} \sum_{y=0}^{q_n-1} z^{\omega_n(y)},$$

where

$$\omega_n(y) = yh_n + s_{n,0} + \ldots + s_{n,y-1}.$$

If $P_n(z)$ are generated by some rank one map, then $P_1(z) \cdots P_n(z)$ always belongs to $\mathcal{M}_{N_n}$.

Generalized Riesz products

The spectral measure σ_f of a function $f \in L^2(X, \mu)$ constant on the levels of a tower with index n_0 is given (up to a constant multiplier) by the infinite product

$$\sigma_f = |\widehat{f}_{(n_0)}|^2 \prod_{n=n_0}^{\infty} |P_n(z)|^2,$$

converging in the weak topology.

Question. Is it possible to construct flat polynomials $P_n(z)$ compatible with some rank one dynamical system?

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Results on singularity

Theorem (Bourgain, 1993). Ornstein rank one transformations have singular spectral type.

Theorem (Klemes, 1994). A class of “staircase” rank one map given by quadratic frequency function $\omega_n(j) = jh_n + j(j-1)/2$ is of singular spectral type.

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Rank one flows

Definition. A *flow* with invariant measure is a family of measure preserving transformations T^t, where $t \in \mathbb{R}$, such that $T^{t+s} = T^t T^s$.

Rank one flows generate a kind of Riesz products

$$|\hat{f}_0(t)|^2 \cdot \prod_{n=1}^{\infty} |P_n(t)|^2$$

with exponential sums as muptipliers

$$P_n(t) = \frac{1}{\sqrt{q_n}}(e^{2\pi i\, t\omega(0)} + e^{2\pi i\, t\omega(1)} + \ldots + e^{2\pi i\, t\omega(q_n-1)}).$$

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Flat exponential sums with coefficients in $\{0, 1\}$

$$\mathcal{M}_q^{\mathbb{R}} = \left\{ \mathcal{P}(t) = \tfrac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i\, t\omega(y)} : \ \omega(y) \in \mathbb{R} \right\}.$$

Theorem. The answer is "yes" in the class $\mathcal{M}_q^{\mathbb{R}}$.
For any $0 < a < b$ and $\varepsilon > 0$ there exists a sum $\mathcal{P}(t) \in \mathcal{M}_q$ which is *compact ε-flat* both in $L^1(a, b)$ and $L^2(a, b)$,

$$\Big\| \big|\mathcal{P}(t)\big|_{(a,b)} \big| - 1 \Big\|_1 < \varepsilon,$$

Flat exponential sums with coefficients in $\{0, 1\}$

and the sums $\mathcal{P}(t)$ are given by the formula

$$\mathcal{P}(t) = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i \, t\omega(y)},$$

where

$$\omega(y) = m\frac{q}{\beta^2} e^{\beta y/q},$$

with appropriate choice of $m > 0$, $\beta^{-1} \in \mathbb{N}$ and q ranging over a set $\mathcal{Q}_{\varepsilon,a,b}(\beta, \varepsilon, m)$ of positive density in $\mathbb{Z}$.

Exponential staircase flow

We construct a rank one flow with the following parameters:

- q_n is the number of subcolumns
- spacers $s_{n,y} = \omega_n(y+1) - \omega_n(y) - h_n$
- $\omega_n(y) = \mu_n \dfrac{q_n}{\beta_n^2} e^{\beta_n y/q_n}, \qquad h_n = \dfrac{\mu_n}{\beta_n}$

$\mu_n \to \infty$ (*slowest*), $\beta_n \to 0$, $q_n \to \infty$ (*fastest*).

Theorem. With certain choice of parameters μ_n, β_n and q_n the rank one flow given by the exponential staircase construction has Lebesgue spectral type.

Exponential sums: Van der Corput's method

Lemma. If $k \to \infty$,

$$\int_a^b e^{ikx^2}\,dx = \sqrt{\frac{\pi}{k}}\exp\left(\frac{2\pi i}{8}\right) + O\left(\frac{1}{k}\right).$$

For real a, $c \neq 0$ and $b > 0$

$$\int_0^b e^{it(a+cx^2)}\,dx = A_0\frac{e^{iat}}{2(|c|t)^{1/2}} - \frac{i}{2bct}e^{it(a+cb^2)} + O\left(\frac{1}{b^3(ct)^2}\right),$$

where $A_0 = e^{2\pi i\,\mathrm{sgn}(c)/8}\sqrt{\pi}$.

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Exponential sums: Van der Corput's method

Let us consider a sum over a interval in the integer line

$$S = \sum_{y=a}^{b} e^{2\pi i f(y)},$$

where $f \in C^2([a, b])$.

Example:

$$f(y) = \frac{y^2}{2(b-a)} + \gamma y + f_0.$$

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Exponential sums: Van der Corput's method

Van der Corput's method – The concept.
We can estimate S as follows:

$$S = \sum_{a<k<\beta} \frac{1}{\sqrt{|f''(y_k)|}} e^{2\pi i(f(y_k) - ky_k + 1/8)} + \mathcal{E},$$

where y_k are solutions of the equation

$$f'(y_k) = k, \qquad \text{where } k \in \mathbb{Z},$$

and $\alpha = f'(a)$, $\beta = f'(b)$.

Points y_k are called *stationary phases* for the function $f(y)$.

Stationary phase dynamics

Our sum $\mathcal{P}(t)$ generates a family of stationary phases $y_k(t)$ depending on t,

$$\mathcal{P}(t) = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i\, t\omega(y)},$$

given by the equation

$$t\omega'(y_k(t)) = k,$$

and the law of evolution for $y_k(t)$ in some cases is expressed by a dynamical system $\dot{y} = v(t, y)$.

Free quantum particle on a compact space

Remark that the sum $\mathcal{P}(t)$ is connected to a quantum dynamical system given on $\mathbb{T}$ by equation

$$i\frac{\partial}{\partial t}\psi = H\psi,$$

where

$$H = \omega\left(-i\frac{\partial}{\partial x}\right),$$

and $\mathcal{P} = \hat{\psi}$.

Stationary phase dynamics: Example

Quadratic function $\omega(y) = H_0(y) = \frac{y^2}{2q}$ generates $y_k(t)$ as follows:

$$t \cdot H_0'(y_k) = k, \quad t \cdot \frac{y_k}{q} = k, \quad y_k(t) = \frac{kq}{t},$$

and the dynamical system induced by H_0 acts on $\mathbb{R}$ as follows:

$$R^t \colon\ x \mapsto \frac{x}{t}, \qquad y_k(t) = R^t y_k(0).$$

Notice that R is an action of the multiplicative group $\mathbb{R}_+$.

Stationary phase dynamics: Idea

Ways of constructing flat $\mathcal{P}(t)$:

- Searching for special unstable cases of arithmetic nature. Example: $2mH_0(y) = my^2/q$, $2m \in 2\mathbb{Z}$ and q is prime.
- Controlling the dynamics of stationary phases $y_k(t)$.

Idea:

(a) The set $\{t\omega(y_k)\}$ is generally chaotic (e.g. for H_0). Could it be constant for some special choice of $\omega(y)$?
or

(b) Is it possible to control the distances: $y_{k+1}(t) - y_k(t) = \text{const}$?

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Stationary phase dynamics: Calculation

Let us suppose that $\frac{d}{dt}(t\omega(y_k)) = 0$, then

$$\omega(y_k) + t\omega'(y_k)\dot{y}_k = 0.$$

Now differentiating the equation $t\omega'(y_k) = k$ we get

$$\omega'(y_k) + t\omega''(y_k)\dot{y}_k = 0,$$

therefore

$$\frac{\partial}{\partial y}\frac{\omega'}{\omega} = 0, \qquad \frac{\omega'}{\omega} = \beta q^{-1} = \mathrm{const},$$

and $\omega(y) = \omega_0 e^{\beta y/q}$.

Stationary phase dynamics for exponential $\omega(y)$

Observe that $\omega(y)$ has expansion (with small parameter β)

$$\omega(y) = \frac{q}{\beta^2} + \frac{y}{\beta} + \frac{y^2}{2q} + \beta\frac{y^3}{6q^2} + \dots$$

Let us associate an $\mathbb{R}_+$-action to our $\omega(y)$. Solving equation

$$t \cdot \omega'(y) = k = \mathrm{const}$$

we have

$$t \cdot \frac{1}{\beta} e^{\beta y/q} = k, \qquad y_k(t) = \frac{q}{\beta} \log \frac{\beta k}{t},$$

Stationary phase dynamics for exponential $\omega(y)$

$$y(t) = y(0) + \frac{q}{\beta} \log t^{-1},$$

and the dynamical system $S^t\colon\ y(0) \mapsto y(t)$,

$$S^t\colon\ x \mapsto x + \frac{q}{\beta} \log t^{-1}$$

acts by translations of the line $\mathbb{R}$.

Dynamical system observation: S^t is much less "chaotic" than R^t.

- R^2 acts on 1-periodic functions as *hyperbolic map*
- and S^t acts on the same space as *rigid rotaion*

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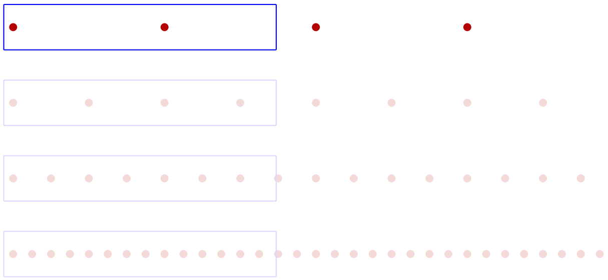
Illustration to the dynamics

Dynamics of R^t



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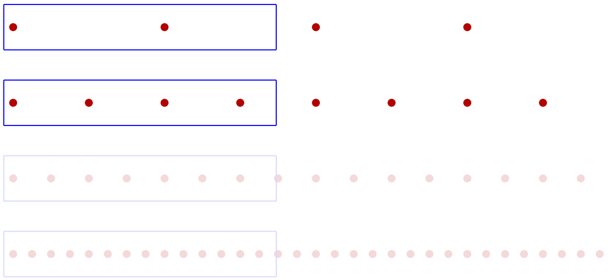

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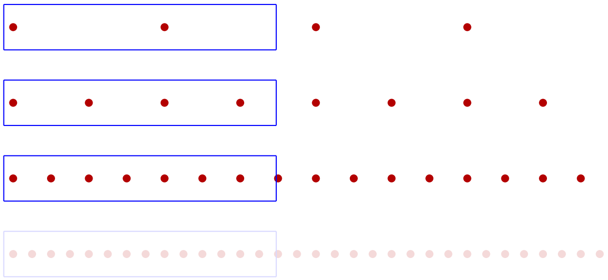

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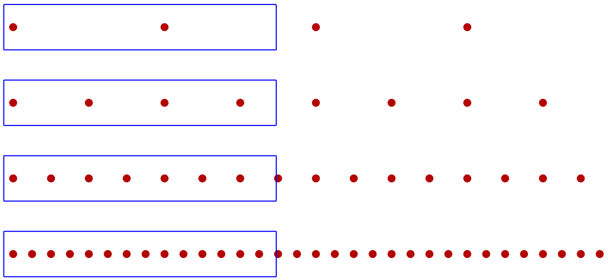


Illustration to the dynamics

Dynamics of S^t

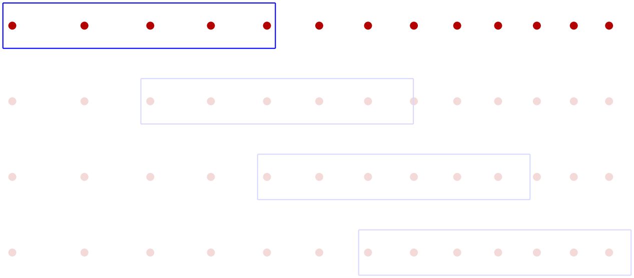


Illustration to the dynamics

Dynamics of S^t

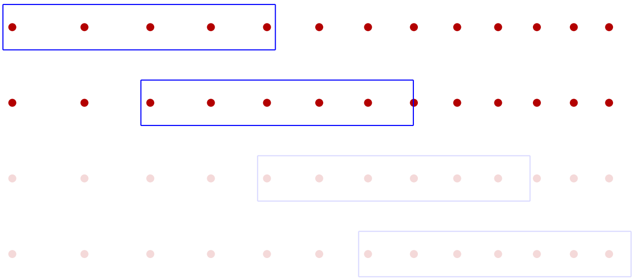
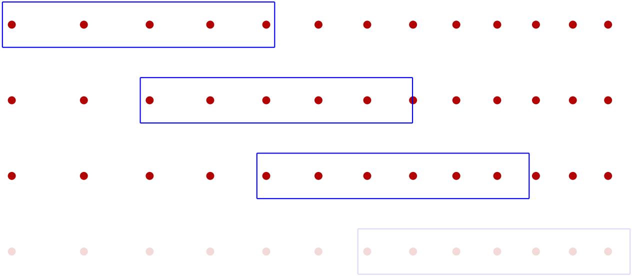


Illustration to the dynamics

Dynamics of S^t



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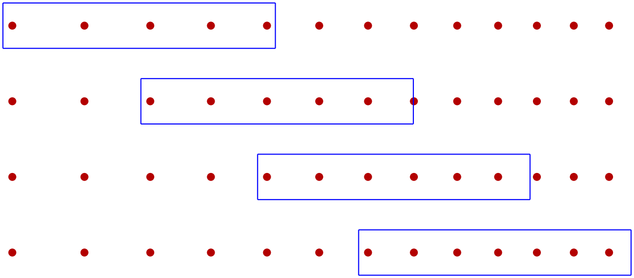
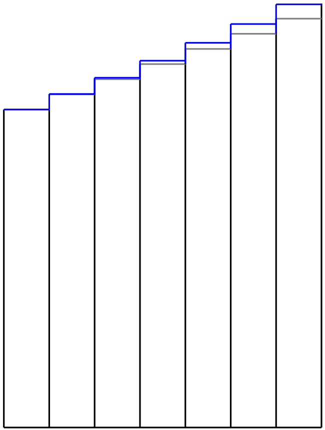


Exponential staircase flow generated by $\omega(y) = \frac{q}{\beta} e^{\beta y/q}$



"...though careful examination had shown that the height of the steps steadily decreased with the rising gravity. The stair had apparently been designed so that the effort required to climb it was more or less constant at every point in its long curving sweep..."

Arthur C. Clarke, *Rendezvous with Rama*

Outline of the proof: Calculation of $y_k(t)$

We see that

$$y_k(t) = \frac{q}{\beta} \log \frac{\beta k}{t},$$

where the index $k \in \mathbb{Z}$ ranges over the interval $(K_0(t), K_1(t))$,

$$K_0(t) = \frac{t}{\beta}, \qquad K_1(t) = \frac{t}{\beta} e^{\beta},$$

$$K_1(t) - K_0(t) \sim t, \qquad \beta \to 0, \quad t \to \infty.$$

Outline of the proof: Van der Corput's method

Applying van der Corput approach we have for $t \to \infty$

$$\mathcal{P}(t) = \frac{1}{\sqrt{t}} \sum_{K_0(t)<k<K_1(t)} e^{2\pi i(t\omega(y_k)-ky_k+1/8)} + \mathcal{E}_1(t).$$

Let us calculate the resulting phase function (minus $1/8$)

$$t\omega(y_k) - ky_k \equiv -ky_k \pmod 1,$$

since

$$t\omega(y_k) = \frac{q}{\beta} \cdot t\omega'(y_k) = \frac{q}{\beta} \cdot k \in \mathbb{Z},$$

if we require that $\beta^{-1} \in \mathbb{Z}$.

Applying van der Corput's method twice

Continuing calculation of the phase function we have

$$-ky_k = -k \cdot \frac{q}{\beta} \log \frac{\beta k}{t} = x(t)k - q \cdot \Omega(k),$$

where

$$x(t) = \frac{q}{\beta} \log \frac{t}{\beta}$$

do not depend on k, and

$$\Omega(k) = \frac{1}{\beta} k \log k$$

do not depend on t.

Applying van der Corput's method twice

Theorem (Poincaré reccurence theorem). Given $\varepsilon > 0$ for a sequence of q of positive density

$$-q \cdot \Omega(k) \approx_{\varepsilon} \Omega(k),$$

for the fixed finite set of $k \in (K_0, K_1) \cap \mathbb{Z}$.

Here we apply the reccurence theorem to the torus shift on $\mathbb{T}^{[t]}$

$$T\colon\ v \mapsto v + \Omega.$$

Applying van der Corput's method twice

It can be easily seen that

$$\Omega'(K_1(t)) - \Omega'(K_0(t)) = 1 + o(1),$$

as $\beta \to 0$ and $t \to \infty$, hence, applying again van der Corput estimate we have

$$\mathcal{P}(t) \approx \frac{1}{\sqrt{t}} \sum_{K_0(t)<k<K_1(t)} e^{2\pi i\,(-ky_k+1/8)} = e^{2\pi i\,\mathcal{A}(t)} + \mathcal{E}_2(t),$$

where $\mathcal{E}_2(t)$ is L^p-small error term for $p = 1$ and $p = 2$.

Scheme of the approach

Exponential sum with coeffitients $\{0, 1\}$

$\rightarrow$

Van der Corput's method (1), reduction: degree q to degree t

$\rightarrow$

Quantum free particle on $\mathbb{T}$

$\rightarrow$

Dynamical system: $\mathbb{R}_+$-action induced by the Hamiltonian $\omega(y)$

$\rightarrow$

Dynamical system on $\mathbb{T}^{[t]}$ given by a torus shift

$\rightarrow$

Van der Corput's method (2), reduction: degree t to degree 1

Thank you!