SPECTRAL THEORY FOR DYNAMICAL SYSTEMS ARISING FROM SUBSTITUTIONS

ANNE SIEGEL

Abstract. Symbolic dynamical systems were first introduced to better understand the dynamics of geometric maps; particularly to study dynamical systems for which past and future are disjoint as for instance toral automorphisms or Pseudo-Anosov diffeomorphisms of surfaces. Self-similar systems are defined to be topologically conjugate to their own first return map on a given subset. A basic idea is that, as soon as self-similarity appears, a substitution is hidden behind the original dynamical system. In this lecture, we first illustrate this idea with concrete examples, and then, try to understand when symbolic codings provide a good representation. A natural queston finally arises: which substitutive dynamical systems are isomorphic to a rotation on a compact group? Partial answers have been given by many authors since the early 60's. Then, we will see how a spectral analysis problem finally reduces to a combinatorial problem, whose partial answers imply Euclidean geometry and even some arithmetics.

1. What is a substitution ?

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^*$ the set of finite words on $\mathcal{A}$. The empty word is denoted ε.

1.1. Substitution on finite words.

A *substitution* or *iterated morphism* is a combinatorial object that simply replaces letters in $\mathcal{A}$ by nonempty finite words. An example on the three-letters alphabet $\mathcal{A} = \{1, 2, 3\}$ is given by σ defined by $1 \mapsto 12$, $2 \mapsto 3$, $3 \mapsto 1$.

As dynamicians, our aim is to iterate this substitution. Hence we formally define a substitution as an endomorphism of the free monoid $\mathcal{A}^*$ endowed with the concatenation (defined by $\sigma(uv) = \sigma(u)\sigma(v)$), such that the image of each letter of $\mathcal{A}$ is nonempty, and such that for at least one letter, say a, the length of the successive iterations $\sigma^n(a)$ tends to the infinity (these two conditions ensure that the substitution can be iterated infinitely).

Then, the successive iterations of the example σ previously defined applied on the letter 1 give

1
12
123
1231
123112
123112123
1231121231231
12311212312311123112
123112123123112311212311212 3
12311212312311231121231123231231121231231

One should notice that for all n, each word $\sigma^n(1)$ starts with the preceeding one $\sigma^n(1)$. Roughly, it is natural to call the infinite iteration of these words a *fixed point* of σ. However, such a fixed point appears to be an infinite word defined as a limit, so that we need to introduce now a couple of formal definitions about infinite sequences and topology.

1.2. **Extension of a substitution to infinite words.** A (finite or infinite) *word* on $\mathcal{A}$ is denoted $w = w_0w_1\dots$. The metrizable topology of the set of infinite words $\mathcal{A}^{\mathbb{N}}$ is the product topology of the discrete topology on each copy of $\mathcal{A}$. A *cylinder* of $\mathcal{A}^{\mathbb{N}}$ is a closed-open set of the form: $[W] = \{(w_i)_i \in \mathcal{A}^{\mathbb{N}} | w_0 \dots w_{|W|-1} = W\}$ for $W \in \mathcal{A}^*$.

A substitution naturally extends by concatenation to the set of infinite words $\mathcal{A}^{\mathbb{N}}$:

$$\sigma(w_0w_1\dots) = \sigma(w_0)\sigma(w_1)\dots$$

A *periodic point* of a substitution σ is an infinite word $u = (u_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ that satisfies $\sigma^{\nu}(u) = u$ for some $\nu > 0$. If $\sigma(u) = u$, then u is a *fixed point* of σ. A simple combinatorial proof states that a substitution may not always have a fixed point, but it always admits at least one periodic point [Que87].

1.3. **What is symbolic dynamics?** A specialist in dynamical systems always looks for maps acting on objects. Dealing with infinite sequences, a natural map immediatly appears, that is, the deletion of the first letter of the word. Formally, we denote by S the *shift map* on $\mathcal{A}^{\mathbb{N}}$ defined by $S(\,(w_i)_{i\in\mathbb{N}}\,) = (w_{i+1})_{i\in\mathbb{N}}$.

Symbolic dynamics consists in studying the shift map on a closed set of infinite sequences of $\mathcal{A}^{\mathbb{N}}$, which is supposed to be invariant through the action of the shift map. We are particularly interested in symbolic sets that are minimal, that is, that do not contain a strictly smaller closed invariant subset.

1.4. **Symbolic dynamical system associated with a substitution.** Dealing with a substitution, a natural process to associate with it a symbolic dynamical consists in first building a fixed point (or a periodic point if a fixed point does not exist) by iteration, then shifting this infinite word infinitely many often (then one gets the *orbit* of the sequence through the shift point), and finally considering the closure of this orbit. However, this process should be interesting provided that when further periodic points do exist, they generate the same symbolic system.

Formally, the *symbolic dynamical system* generated by a word u is the pair (X_u, S), where X_u denotes the closure in $\mathcal{A}^{\mathbb{N}}$ of the orbit $\{S^n u,\, n \in \mathbb{N}\}$ of u under the shift map. The shift map S is an homeomorphism on this compact subset of $\mathcal{A}^{\mathbb{N}}$.

We call a substitution σ *primitive* if there exists an integer ν (independent of the letters) such that, for each pair $(a, b) \in \mathcal{A}^2$, the word $\sigma^{\nu}(a)$ contains at least one occurrence of the letter b.

Theorem 1.1 (see [Que87, PF02]). *Let σ be a primitive substitution. If u is a periodic point for σ, then X_u does not depend on u and we denote by (X_σ, S) the* symbolic dynamical system generated *by σ. The system (X_σ, S) is* minimal *and* uniquely ergodic*: X_σ contains no non-empty closed shift-invariant subset and there exists a unique shift-invariant probability measure μ_{X_σ} on X_σ.*

Notice that the property of minimality has a combinatorial interpretation in this case: (X_σ, S) is minimal if and only if any every word occurring in a periodic point u appears in an infinite number of positions with bounded gaps.

2. From geometric dynamics to symbolic dynamics

Historically, symbolic dynamics has been introduced to better understand the dynamics of geometric maps. Indeed, by coding the orbits of a dynamical system with respect to a cleverly chosen finite partition indexed by the alphabet $\mathcal{A}$, one can replace the initial dynamical system, which may be difficult to understand, by a simpler dynamical system, that is, the shift map on a subset of $\mathcal{A}^{\mathbb{N}}$.

This old idea was used intensively, up to these days, particularly to study dynamical systems for which past and future are disjoint, such as toral automorphisms or pseudo-Anosov diffeomorphisms of surfaces. These systems with no memory, whose entropy is strictly positive, are coded by subshifts of finite type, defined by a finite number of forbidden words, and belong to the *Markov* framework. Some very important literature has been devoted to their many properties (see [LM95]). The partitions which provide a good description for a topological dynamical system, leading to a subshift of finite type, are called *Markov partitions.*

2.1. **An example of the use of symbolic dynamics: The Morse sequence.** In 1920, M. Morse was studying *geodesics*, that is, the curves realizing the minimum distance between two points, on connected surfaces with constant negative curvature. He was looking at infinite geodesics which remain within a small part of the space. More precisely, a geodesics is said to be *recurrent* if every point of the geodesics lies at a given distance (whatever small it can be) of a point in every long enought segment of the geodesics. Hence, closed geodesics are recurrent or periodic. An intricate question is the existence of non-closed recurrent geodesics.

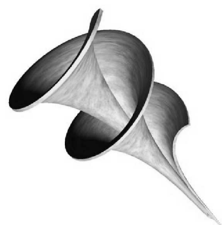


FIGURE 1. Two examples of connected surfaces with constant negative curvature.

To answer this question, in [Mor21], using a method initiated by Hadamard, Morse did a *coding* of geodesics, by infinite sequences of 0's and 1's, according to which boundary of the surface they meet: thus, we arrive in the space $\{0,1\}^{\mathbb{N}}$ of infinite symbolic sequences. To advance along a geodesic translates into looking at the next element of the sequence. The coding sends under suitable conditions the topology of the surface onto the product topology in $\{0,1\}^{\mathbb{N}}$.

Properties of geodesics are then easy to check: a closed geodesic corresponds to a periodic sequence. In the same way, by replacing points by elementary segments, the reader shall be able to check that a recurrent geodesic corresponds to what is now called a *minimal* sequence: every word occurring in u appears an infinite number of positions with bounded gaps.

Thanks to this coding, Morse proved the existence of a closed and recurrent geodesics:

Theorem 2.1 ([Mor21]). *A minimal and nonperiodic sequence is given by the (Prouhet-Thue)-Morse sequence,*

$$0110100110010110100101100110100110010110011010010110100110010110 1\ldots$$

defined as the fixed point (starting with 0*) of the Morse substitution* $\sigma : 0 \mapsto 01 \quad 1 \mapsto 10$.

A full study of the Morse sequence is made in [PF02].

2.2. **Self-similar dynamics and substitutions.** Dealing with a dynamical system, a usual problem is to try to understand the local structure of its orbits. A classical method to study this problem is to consider the *first return map* (Poincaré map) over an appropriate neighborhood $\mathcal{N}$ of a given point. For some systems such as toral quadratic rotations or some interval exchanges with parameters living in a quadratic extension, the system defined by the first return map on some subset is topologically conjugated to the original system. One can say that the original dynamical system has a *self-similar structure.* A basic idea is that, in general, as soon as self-similarity occurs, a substitution is hidden behind the original dynamical system: the trajectories of points in the neighborhood $\mathcal{N}$ before they come back into $\mathcal{N}$, define a substitution. Then, the trajectories of the points of the full system belong to the symbolic dynamical system associated with the substitution. Let us immediatly illustrate this idea with a simple example.

2.3. Example: addition of the golden ratio. Let φ denote the *addition of the golden ratio* $\alpha = 1 - \alpha^2 = 1,61...$ on the one-dimensional torus $\mathbb{T}$:

$$\varphi : x \in \mathbb{T} = \mathbb{R}\backslash\mathbb{Z} \quad \mapsto \quad x + \alpha \bmod 1 \in \mathbb{T}.$$

This map has two intervals of continuity:

$$\mathbb{T} = I_2 \cup I_1, \text{ with } I_2 = [0, 1-\alpha[, \quad I_1 = [1-\alpha, 1[.$$

Let ψ denote the first return map of φ on the largest interval of continuity I_1, that is,

$$\forall\, x \in [1-\alpha, 1[, \quad \psi(x) = \varphi^{\min\{k \in \mathbb{N}^*,\, \varphi(x) \in [1-\alpha,1[\}}(x).$$

We are going to prove thanks to a short computation that ψ is equal to φ itself, up to a reversal of the orientation and a renormalization.

Indeed, Let us consider the following partition of I_1:

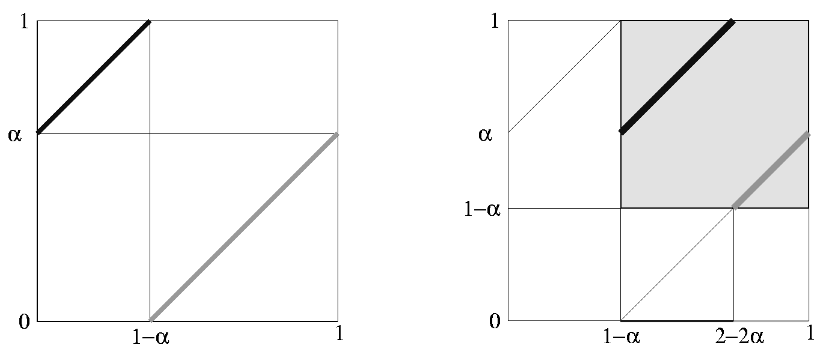
$$I_1 = J_1 \cup J_2, \text{ with } J_1 = [1-\alpha, 2-2\alpha[, \quad J_2 = [2-2\alpha, 1[.$$

Then a simple computation yields that ψ restricted to J_1 is equal to φ^2:

- $J_1 = [1-\alpha, 2-2\alpha[\subset I_1$,
- $\varphi(J_1) = [0, 1-\alpha[\not\subset I_1$,
- $\varphi^2(J_1) = [\alpha, 1[\subset I_1$.

Similarly, since $J_2 = [2-2\alpha, 1[\subset I_1$ and $\varphi(J_1) = [1-\alpha, \alpha[\subset I_1$, ψ restricted to J_2 is equal to φ.

A graphical representation of φ and ψ is given in Figure 2: the two graphics appear to be equal up to a reversal of the orientation and a renormalization. Formally, there is no difficulty to prove that φ and ψ are homeomorphic through the conjugacy map $\tau : x \in [0, 1[\mapsto (1-\alpha)x + 1 \in [1-\alpha, 1[$.



Representation of φ First return map ψ on $[1-\alpha, 1[$

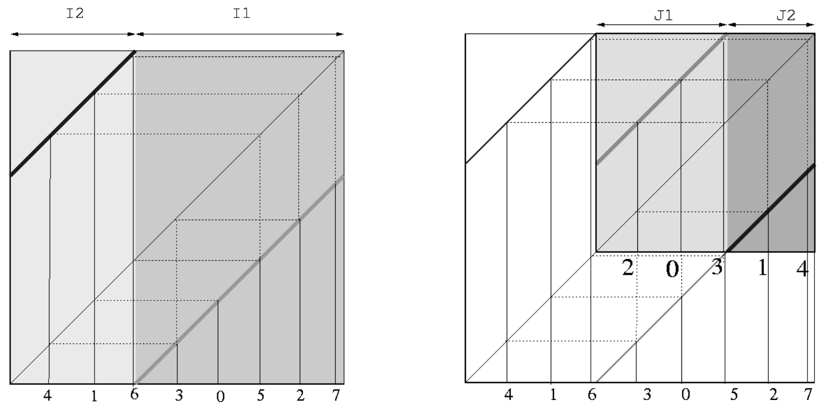
FIGURE 2. The addition of the golden ratio is equal to its first return map up to a reversal of the orientation and a renormalization.

The interest of such a coding is that we are now able to code the trajectories of a point in $[1-\alpha, 1[$ for both the addition φ of the golden ratio and its first return map ψ. Let us study the example shown in Fig. 3. Indeed, the point α mod 1 (denoted by 0 on each figure) belongs to the largest interval I_1. Then one sees that $\varphi(\alpha)$ (denoted by 1) belongs to I_2, $\varphi^2(\alpha) \in I_1$, etc. Then the trajectory of α is coded by $I_1 I_2 I_1 I_1 I_2 I_1 I_2 I_1$.

Similarly, computing the trajectory of α for the first return map ψ gives $J_1 J_2 J_1 J_1 J_2$.

The main point is that, since ψ is defined as the first return map of φ, there is a relationship between the two codings introduced here. Indeed, as soon as a point x lies in J_1, then we know that

- x belongs to I_1,
- $\varphi(x) \in I_2$
- $\psi(x) = \varphi^2(x)$.



Trajectory through φ:
$I_1I_2I_1I_1I_2I_1I_2I_1$.

Trajectory through ψ:
$J_1J_2J_1J_1J_2$

FIGURE 3. Trajectories of the point α relatively to intervals of continuity of φ and its first return map ψ

Hence, coding a point x by J_1 according to ψ implies that the trajectory of the same point will be coded by I_1I_2 according to φ. Similarly, coding a point x by J_2 according to ψ implies that the trajectory of the same point will be coded by I_1 according to φ. We thus deduce that the trajectory of a point x through φ can be obtained by mapping the trajectory of a point x through ψ thanks to the map:

$$J_1 \to I_1I_2; \quad J_2 \to I_1.$$

One should remember now that we stated that φ and ψ were conjugate through the map τ. However, α is a fixed point for τ, and the partition $J_1 \cup J_2$ is the image of $I_1 \cup I_2$ through τ. Hence, the trajectories of α have the same coding through φ and ψ. Consequently, this coding must be nothing else than the fixed point of the following substitution, called the Fibonacci substitution

$$1 \mapsto 12; \quad 2 \mapsto 1.$$

One finally proves that the addition of the golden ratio is very well represented as a symbolic shift map:

Theorem 2.2 (see a general proof in [AFH99]). *The coding of the trajectory of α mod 1 through the addition φ by the golden ratio α on $\mathbb{T}$ according to the intervals of continuity I_1 and I_2 is the fixed point of the Fibonacci substitution $1 \to 12$, $2 \to 1$:*

$$u = 121121211211212112121121121211211212112121121121211212112112121...$$

The set of codings of all the points of $\mathbb{T}$ is equal the symbolic dynamical system associated with the Fibonacci substitution. The coding map is a semi-topological conjugacy between the shift map on the symbolic system and the addition by the golden ratio.

Remark 2.3. For the example of the toral addition by the golden ratio, we can define an inverse map, from the symbolic system onto the torus. It is proved that this map is continuous, 2-to-1, and 1-to-1 except on a countable set; this is the best possible result, given the fact that one of the sets is connected and the other one a Cantor set.

3. From symbolic dynamics to geometry ?

As shown in Section 2.3, Poicaré's method defines a coding map from the geometric system onto the substitutive symbolic dynamical system. A natural question is: how far is this map from being a bijection? We have just seen that a precise answer has been given to this question for the

Fibonacci substitution (Remark 2.3). For other examples, the question can be much more difficult. It is natural then to focus on the reverse question: given a substitution, which self-similar geometric actions are coded by this substitution?

For the Morse substitution, it is proved that the symbolic dynamical system associated with this substitution is a two-point extension of the dyadic odometer, that is, the group $\mathbb{Z}_2$ of 2-adic integers ([dJ77] and also [PF02], chapter 2).

The three-letter equivalent of the Fibonacci substitution is the Tribonacci substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. G. Rauzy, with methods from number theory, proved in 1981 that the symbolic dynamical system associated with this substitution is measure-theoretically isomorphic, by a continuous map, to a domain exchange on a self-similar compact subset of $\mathbb{R}^2$ called the *Rauzy fractal* [Rau82]. Tiling properties of the Rauzy fractal yield an isomorphism between the substitutive system and a translation on the two-dimensional torus. This example will be studied in more details in Section 3.2.

These examples emphasize the connection between searching for a geometric interpretation of a symbolic dynamical systems and understanding whether this dynamical system is already known up to an isomorphism. Since substitutive dynamical systems are deterministic, i.e., of zero entropy, they are very different from subshifts of finite type. Hence, the following question is natural: which substitutive dynamical systems are isomorphic to a translation on a compact group? More generally, what is their maximal equicontinuous factor, that is, the largest translation on a compact group that topologically embeds into this symbolic system?

Let us introduce now the point of view of spectral theory. Indeed, to a dynamical system (X, S) is associated the unitary operator $U: f \in L^2(X_\sigma, S) \mapsto f \circ S \in L^2(X_\sigma, S)$ [Wal82]. One usual calls *eigenvalues* of the dynamical system the eigenvalues λ's of U; their modulus is equal to one, so that the word *eigenvalue* sometimes also holds for every $x \in [0,1[$ such that $\lambda = e^{2i\pi x}$. The *eigenfunctions* of the dynamical system are the eigenfunctions of U; they appear to be functions $f \in L^2(X_\sigma, S)$.

From this point of view, the maximum equicontinuous factor of a dynamical system is proved to be the unique abelian compact group translation with the same eigenvalues than the dynamical system. Hence, it uniquely determined by the eigenvalues [Wal82].

Starting from a geometrical and combinatorial question, we naturally come to a question of spectral theory, that is, computing the eigenvalues of a dynamical system.

3.1. **Substitution of constant length.** During the seventies, a precise answer to this question has been obtained for substitutions of constant length (the images of each letters in the alphabet share the same length) [Kam70, Mar71, Dek78]. This caracterization implies some p-adic groups $\mathbb{Z}_p$, also called p-adic odometer, obtained as the completion of $\mathbb{Z}$ for the p-adic topology [Gou97].

Theorem 3.1 (Dekking [Dek78]). *Let σ be a substitution of constant length n. Let $u = (u_n)_{n\in\mathbb{N}}$ be a periodic point for σ. We call* height *of the substitution the greatest integer m which is coprime with n and divides all the strictly positive ranks of occurrence of the letter u_0 in u. The height is less that the cardinality of the alphabet.*

The maximal equicontinuous factor of the substitutive dynamical system associated with σ is the addition of $(1,1)$ on the abelian group $\mathbb{Z}_n \times \mathbb{Z}/m\mathbb{Z}$, where $\mathbb{Z}_n$ denotes the product of the p-adic groups $\mathbb{Z}_p$ for every prime p that divides n.

As an example, the letter 1 appears at rank 3 and 5 in the fixed point

$$u = 122\mathbf{1}2\mathbf{1}1221121221211 21...$$

of the Morse substitution so that this sustitution has height 1. Hence, the maximal equicontinuous factor of the associated substitutive system is the 2-adic group $\mathbb{Z}_2$.

An example of a substitution with an height different from 1 is given by $1 \mapsto 121 \quad 2 \mapsto 312 \quad 3 \mapsto 213$: the letter 1 appears at every even rank in the fixed point

$$u = 12\mathbf{1}3\mathbf{1}2\mathbf{1}2\mathbf{1}213121312121312$$

so that the height is 2 and the maximal equicontinuous factor is $\mathbb{Z}_3 \times \mathbb{Z}/2\mathbb{Z}$.

Dekking also provides a necessary and sufficient condition for a measure-theoretic isomorphism between such a substitutive system of constant length and its maximal equicontinuous factor. This condition is purely combinatorial: a substitution σ is said to satisfy the *coincidence condition* if there exists n such that the image of each letter under a power σ^k has the same n-th letter. We have:

Theorem 3.2 (Dekking [Dek78]). *Let σ be a substitution of constant length and of height 1. The substitutive dynamical system associated with σ has a purely discrete spectrum if and only if the substitution σ satisfies the condition of coincidence.*

As an example, the substitution $1 \mapsto 12 \quad 2 \mapsto 23 \quad 3 \mapsto 13$ has a pure discrete spectrum dynamical system since its three fixed points contain a 1 at rank 6:

$$\begin{array}{c} 122323\mathbf{1}23131213231312131\ldots \\ 231312\mathbf{1}31223121312232313\ldots \\ 122323\mathbf{1}32313121312232313\ldots \end{array}$$

Conversely, the two fixed points of the Morse substitution have no coincidence so that the associated dynamical system is not isomorphic to the dyadic odometer.

In the case when the height of the substitution is different from 1, it is possible to recode the substitution into a substitution with height 1 and to check the coincidence condition on this last substitution. As an example of application, this allows one to prove that the substitution $1 \mapsto 121 \quad 2 \mapsto 312 \quad 3 \mapsto 213$ introduced previously has a pure discrete spectrum dynamical system (see [Dek78] and [PF02], Chap. 7 for details).

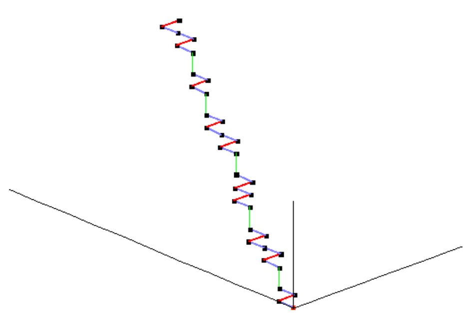
3.2. **A first step towards the study of substitution of nonconstant length: The Tribonacci substitution.** G. Rauzy generalized in [Rau82] the dynamical properties of the Fibonacci substitution to a three-letter alphabet substitution, called the Tribonacci substitution or Rauzy substitution, and defined by

$$\sigma(1) = 12 \qquad \sigma(2) = 13 \qquad \sigma(3) = 1.$$

Broken line associated with the substitution – Let $u =$ denote the unique infinite fixed point of σ:

$$u = 1213121121312121312112131213121121312121 3121\ldots$$

Let us embed this infinite word u as a broken line in $\mathbb{R}^3$ by replacing succesively each letter of u by the corresponding vector in the canonical basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in $\mathbb{R}^3$.



An interesting property of this broken line is that it remains at a bounded distance of a line, turning around it. One states that this axis if nothing else that the expanding direction of the incidence matrix of the substitution, that is, the matrix that contains in each column j the number of occurences of each letter i in $\sigma(j)$.

$$\mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notice that the reason why σ is called the *Tribonacci substitution* is that the characteristic polynomial of $\mathbf{M}_\sigma$ is $X^3 - X^2 - X - 1$ so that its roots satisfy $\alpha^3 = \alpha^2 + \alpha + 1$, and are called Tribonacci numbers in reference to the Fibonacci number. One root is strictly greater than 1 and is associated with an expanding eigenline; the two other roots are complex conjugates of modulus less than 1. They generate a contracting plane.

Definition of the Rauzy fractal – When one projects the vertices of the broken line onto the contracting plane of $\mathbf{M}_\sigma$, along the expanding direction, then one obtains a bounded set in a two-dimensional vector space. The closure of this set of points is a compact set denoted by $\mathcal{R}$ and called the *Rauzy fractal* (see Fig. 4).

To be more precise, denote by π the linear projection in $\mathbb{R}^3$, parallel to the expanding direction of $\mathbf{M}_\sigma$, on the contracting plane of $\mathbf{M}_\sigma$, identified with the complex plane $\mathbb{C}$. If $u = (u_i)_{i\in\mathbb{Z}}$ is the periodic point of the substitution, then the Rauzy fractal is

$$\mathcal{R} = \overline{\left\{ \pi\left(\sum_{i=0}^{n} \mathbf{e}_{u_i}\right) ; \; n \in \mathbb{Z} \right\}}.$$

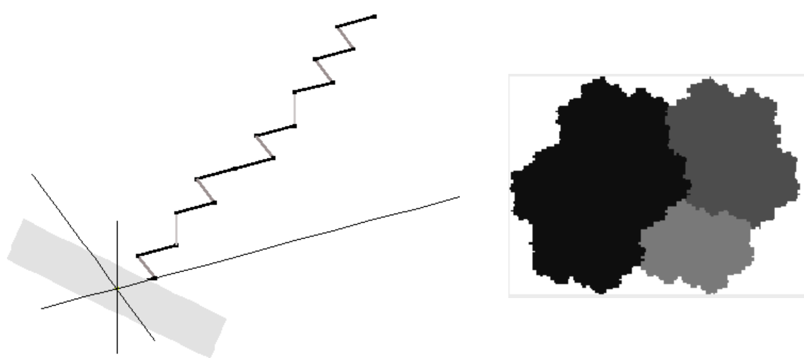


FIGURE 4. The projection method to get the Rauzy fractal for the Tribonnacci substitution.

Partition of the Rauzy fractal – As shown in Fig.4, three subsets of the Rauzy fractal can be distinguished. Indeed, for each letter $j = 1, 2, 3$, the *cylinder* $\mathcal{R}_j$ is defined to be the closure of the set of ends of any segment on the broken line which is parallel to the canonical vector $\mathbf{e}_j$:

$$\mathcal{R}_j = \overline{\left\{ \pi\left(\sum_{i=0}^{n} \mathbf{e}_{u_i}\right) ; \; n \in \mathbb{Z}, \; u_{n+1} = j \right\}}.$$

The union of these three cylinders covers the compact $\mathcal{R}$, and G. Rauzy proved in [Rau82] that their intersection has zero measure.

Dynamics on the Rauzy fractal – One should notice that it is possible to move on the broken line, from a vertex to the following one, thanks to a translation by one of the three canonical vectors $\mathbf{e}_1$,

$\mathbf{e}_2$ or $\mathbf{e}_3$. In the contracting plane, this means that each cylinder $\mathcal{R}_i$ can be translated by a given vector, i.e., $\pi(\mathbf{e}_i)$, without going out of the Rauzy fractal: $\mathcal{R}_i + \pi(\mathbf{e}_i) \subset \mathcal{R}$.

Thus, the following map φ, called a *domain exchange* (see Fig. 5) is well defined for any point of the Rauzy fractal which belongs to only one set $\mathcal{R}_j$. Since the cylinders intersect on a set of measure zero, this map is defined almost everywhere on the Rauzy fractal:

$$\forall x \in \mathcal{R}, \quad \varphi(x) = x + \pi(\mathbf{e}_i), \quad \text{if } x \in \mathcal{R}_i.$$

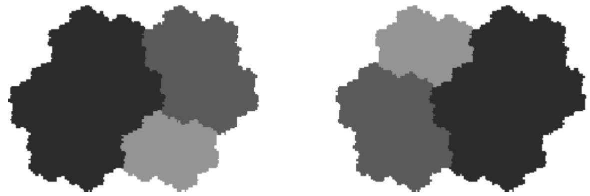


FIGURE 5. Domain exchange over the Rauzy fractal.

It is natural to code, up to the partition defined by the 3 cylinders, the action of the domain exchange φ over the Rauzy fractal $\mathcal{R}$. G. Rauzy proved in [Rau82] that the coding map, from $\mathcal{R}$ into the three-letter alphabet full shift $\{1,2,3\}^{\mathbb{Z}}$ is almost everywhere one-to-one. Moreover, this coding map is onto the substitutive system associated with the Tribonacci substitution. Thus we have the following result:

Theorem 3.3 (Rauzy, [Rau82]). *The domain exchange φ defined on the Rauzy fractal $\mathcal{R}$ is semi-topologically conjugate to the shift map on the symbolic dynamical system associated with the Tribonacci substitution.*

Factorization onto a torus – The domain exchange φ is defined only almost everywhere, which prevents us to define a continuous dynamics on the Rauzy fractal. A solution to this problem consists in factorizing the Rauzy fractal by the lattice $\mathcal{L} = \mathbb{Z}\,\pi(\mathbf{e}_1 - \mathbf{e}_3) + \mathbb{Z}\,\pi(\mathbf{e}_2 - \mathbf{e}_3)$. Indeed, this quotient map sends the contracting plane onto a two-dimensional torus; the three vectors $\pi(\mathbf{e}_1)$, $\pi(\mathbf{e}_2)$ and $\pi(\mathbf{e}_3)$ map onto the same vector on the torus. Thus, the factorization of the domain exchange φ on the quotient is a toral translation.

G. Rauzy proved in [Rau82] that the restriction of the quotient map to the Rauzy fractal is onto and almost everywhere one-to-one. Consequently, we get that the domain exchange on the Rauzy fractal, which is known to be semi-topologically conjugate to the Tribonacci substitutive dynamical system, is also measure-theoretically isomorphic to a minimal translation on the two-dimensional torus $\mathbb{T}^2$. Finally, by mixing dynamics, self-similarity and number theory, we get the two following equivalent results:

Theorem 3.4 (Rauzy, [Rau82]). *The symbolic dynamical system generated by the Tribonacci substitution is measure-theoretically isomorphic to a toral translation, that is, it has a purely discrete spectrum.*

3.3. **A significant advance towards the understanding of the spectrum of substitutive systems.** B. Host made a significant contribution to the understanding of ergodic properties of substitutive systems; in [Hos86], any class of eigenfunctions is proved to contain a continuous eigenfunction (see [Wal82] for usual definition about ergodic theory). Thus, the two main dynamical classifications (up to measure-theoretic isomorphism and topological conjugacy) are equivalent for primitive substitutive systems.

Coboundaries – In the continuation of this, the notion of *coboundaries* introduced by B. Host allows one to better understand the structure of the spectrum of a substitutive system. A *coboundary* of a substitution σ is defined as a map $h : \mathcal{A} \to \mathbb{U}$ (where $\mathbb{U}$ denotes the unit circle) such that there exists a map $f : \mathcal{A} \to \mathbb{U}$ with $f(b) = f(a)h(a)$ for every word ab of length 2 that appears in a periodic point for σ. The coboundary defined by $h(a) = 1$ for every letter a (that is, $f(a) = f(b)$ for every ab in the language) is called the *trivial coboundary.* For substitutions of constant length, nontrivial coboundaries are related to the finite group contained in the maximal equicontinuous factor described in Theorem 3.1. Details can be found in [PF02], Chap. 7.

In the most simplest cases the only coboundary is the trivial one, that is, the constant function equal to 1. However, there exist some substitutions with nontrivial coboundaries such as $1 \mapsto 1231$, $2 \mapsto 232$, $3 \mapsto 3123$. Indeed, words of length 2 that appear in the fixed point of this substitution begining with 1 are 12, 23, 31 and 32. Hence for every $\lambda \in [0, 1[$, the function $h(1) = 1$, $h(2) = e^{2\pi\lambda} = 1/h(3)$ defines a non-trivial couboundary associated with the function $f(1) = 1 = f(2)$, $f(3) = e^{2\pi\lambda}$.

Structure of the spectrum – Coboundaries allow Host to describe precisely the structure of the spectrum defined in Section 3.

Theorem 3.5 (Host [Hos86]). *Let σ be a primitive substitution over the alphabet $\mathcal{A}$. A complex number $\lambda \subset \mathbb{U}$ is an eigenvalue of (X_σ, S) if and only if there exists $p > 0$ such that for every $a \in \mathcal{A}$, the limit $h(a) = \lim_{n\to\infty} \lambda^{|\sigma^{pn}(a)|}$ is well defined, and h is a coboundary of σ.*

Hence, the spectrum of a substitutive system can be divided into two parts.

Arithmetic spectrum: incidence matrix – Since the constant function equal to one 1 is always a coboundary, a sufficient condition is the following: if there exists $p \in \mathbb{N}$ such that $\lambda \in \mathbb{C}$ satisfies $\lim \lambda^{|\sigma^{pn}(a)|} = 1$ for every letter a of the alphabet, then λ is an eigenvalue of the substitutive dynamical system associated with σ.

Such eigenvalues are said to be *arithmetic* since they are computable (the condition $\lim \lambda^{|\sigma^{pn}(a)|} = 1$ can be interpreted in terms of scalar product) and depend only on the incidence matrix of the substitution. Especially, two substitutions that differ only by the order of occurencies of the letters in images of the letters have the same arithmetical spectrum (see [PF02], Chapter 7).

Combinatorial spectrum: return words – Conversely, the eigenvalues for non-trivial coboundaries are "non-commutative": they depend heavily on the combinatorics of the substitution. Durand [Dur98], Ferenczi [FMN96] and Livshits [Liv87] established that they depend on *return words*, playing the role of the height that was defined for substitutions of constant length. Roughly, a *return word* is a word $W = a_1 \dots a_k$ such that Wa_1 is in a factor of the periodic point of the substitution, and $a_i \neq a_1$ for all i. A more precise definition should be found in [PF02].

A condition for no combinatorial spectrum: coincidences – A combinatorial condition is related to the existence of only a trivial coboundary. This condition is called strong coincidence condition and generalizes the condition of Dekking. It was defined by Host, Hollander and formalized by Arnoux and Ito [AI01]. Formally, σ is said to satisfy the *strong coincidences conditions* if for every pair of letters b_1, b_2, there exists a letter a and $P_1, S_1, P_2, S_2 \in \mathcal{A}^*$ such that

$$\sigma^n(b_1) = P_1\, a\, S_1 \quad \sigma^n(b_2) = P_2\, a\, S_2.$$

Coincidences are related to coboundaries by the following result (see a proof in [PF02], Chapter 7).

Lemma 3.6 (Host). *Let σ be a substitution with a nontrivial coboundary $g : \mathcal{A} \to \mathbb{U}$. Let f be the function of modulus 1 which satisfies $f(b) = g(a)f(a)$ as soon as the word ab belongs to the language of a periodic point of the substitution. If there exist two letters a and b and a rank k such that*

- *$f(a) \neq f(b)$,*

- *$\sigma^k(a)$ begins with a and $\sigma^k(b)$ begins with b,*

then σ does not satisfy the coincidence condition on prefixes.

Roughly, this lemma means that a substitutive system with coincidences do not have a combinatorial spectrum. However, we are unable to prove this last result in general, but only for substitutions of Pisot type (see Section 4.2).

4. Applications

4.1. **Properties of the spectrum of substitutive systems.** From the end of the 80's, many papers have provided conditions for a substitutive dynamical system to have a purely discrete spectrum [Liv87, VL92, Sol92, HS03]. Some are necessary conditions, others are sufficient conditions. Let us focus on some typical examples of applications.

- Weakly mixing examples of substitutive systems are derived From Host's results, as $1 \mapsto 12121$, $2 \mapsto 112$, since 1 is the only eigenvalue of the associated substitutive system.
- Refinements of Host's theory allowed Livshits to define conditions for pure discrete spectrum or partially continuous spectrum, as a mix of the coincidence condition and return words. Hence, the system associated with $1 \mapsto 23$, $2 \mapsto 12$, $3 \mapsto 23$, has as a continuous spectral component but is not weakly mixing [Liv87, VL92].
- An important result is stated by Solomyak in the case when the incidence polynomial of a substitution is irreducible: the existence of discrete spectrum depends on the expanding eigenvalues of the incidence matrix of the substitution. Indeed, if there exist $P \in \mathbb{Z}[X]$ and $C \in \mathbb{R}$ such that $P(\alpha) = C$ for every expanding eigenvalue α of the matrix, then $exp(2\pi i C)$ is an eigenvalue of (X_σ, S) [Sol92]. A partial converse was established by Ferenczi, Mauduit, Nogueira [FMN96]. This allows one to compute explicitly the spectrum of some substitutive systems, such as $1 \mapsto 1244$, $2 \mapsto 23$, $3 \mapsto 4$, $4 \mapsto 1$, whose spectrum is $exp(2\pi i\mathbb{Z}\sqrt{2})$.

4.2. **A specific class of substitutions: substitutions of Pisot type.** A substitution σ is *of Pisot type* if every non-dominant eigenvalue λ of its incidence matrix $\mathbf{M}$ satisfies $0 < |\lambda| < 1$. We deduce that the characteristic polynomial of the incidence matrix of such a substitution is irreducible over $\mathbb{Q}$. Consequently, the dominant eigenvalue α is a Pisot number and the other eigenvalues λ are its algebraic conjugates and substitutions of Pisot type are primitive (see the proofs in [PF02]). A substitution σ is *unimodular* if $\det \mathbf{M} = \pm 1$.

The spectrum of substitutive systems of Pisot type has some important properties:

- such systems are never weakly mixing since they have only one expanding eigenvalue so that they satisfy the conditions of Solomyak given in Section 4.1.
- Their arithmetical spectrum can be computed thanks to Host's method. In the unimodular case, the arithmetic spectrum is generated by the frequencies of the letters in the fixed point. In the non-unimodular case, additional rational eigenvalues have to be computed.
- Substitutions of Pisot type never has a nontrivial coboundary [BK04]. Hence, their spectrum is equal to their arithmetic spectrum, which is explicit as explained in the preceeding item.

From these properties, one naturally wonders whether substitutions of Pisot type have a pure discrete spectrum. Unfortunately, a positive answer is not so easy to give.

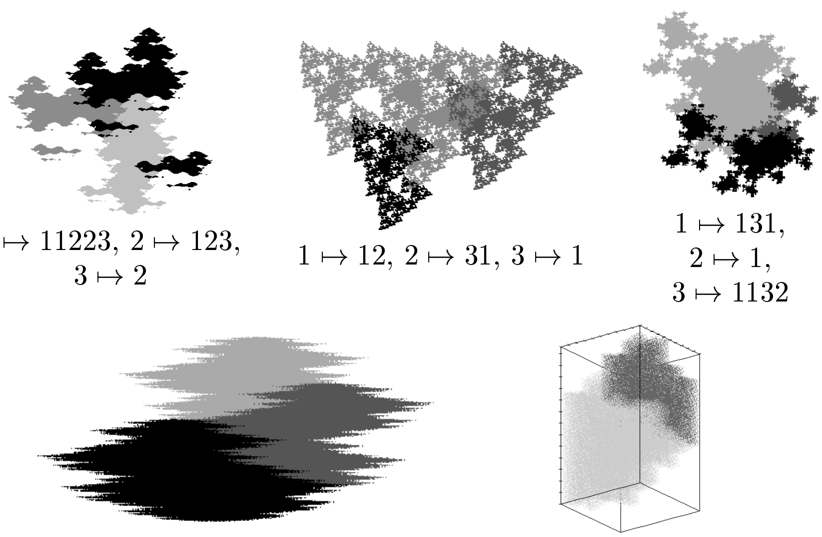
The case of substitutions on a two-letters alphabet is completely studied. We first know from the work of Host and Solomyak-Hollander that substitutions that are of of Pisot type with coincidences on two letters all have a pure discrete spectrum dynamical system [HS03]. Then, Barge and Diamond proved that substitutions of Pisot type on two letters always have coincidences [BD02]. This yields the following theorem:

Theorem 4.1. *All substitutive systems of Pisot type on two letters have a pure discrete spectrum.*

However, on more than three letters, the methods used before are not successful anymore. More intricate results have to be proved in the flavour of Rauzy's work for the Tribonacci substitution.

4.3. **Rauzy fractals.** Starting for a substitution of Pisot type, nothing prevents one from computing a Rauzy fractal as done for the Tribonacci substitution:

1. one can build a broken line from a periodic point of the substitution. Since the substitution is of Pisot type, the broken line turns around a one-dimensional direction and projects onto a compact set called the *Rauzy fractal* of the substitution. If the substitution is not unimodular, then the projection space should take into account an arithmetic part. More precisely, the space of projection is a product of the Euclidean space with finite extensions of p-adic spaces that has a non-zero Haar measure [Sie03].
2. A piece on the Rauzy fractal is associated with each letter of the alphabet. The strong coincidence condition means that the pieces are disjoint in measure [AI01]. Finally, the *Rauzy fractal* of a Pisot type substitution with strong coincidences appears to be *self-similar* and compact.
3. Shifting the fixed point, that is moving on the broken line, factorize onto an exchange of domains on the Rauzy fractal. Arnoux and Ito proved that the shift map and the domain exchange are equivalent from a spectral point of view, as stated in Theorem 4.2.



$1 \mapsto 11223,\ 2 \mapsto 123,\ 3 \mapsto 2$

$1 \mapsto 12,\ 2 \mapsto 31,\ 3 \mapsto 1$

$1 \mapsto 131,\ 2 \mapsto 1,\ 3 \mapsto 1132$

$1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 132$

$1 \mapsto 1112,\ 2 \mapsto 12$

FIGURE 6. Example of Rauzy fractals for substitutions of Pisot type.

Theorem 4.2. *Let σ be substitution of Pisot type over a d-letter alphabet which satisfies the condition of coincidence. Then the substitutive dynamical system associated with σ is measure-theoretically isomorphic to the exchange of d domains defined almost everywhere on the Rauzy fractal of σ, that is, a self-similar compact set on a product of the Euclidean space with finite extensions of p-adic spaces that has a non-zero Haar measure.*

Notice that we do not know any example of a substitution of Pisot type with no strong coincidence.

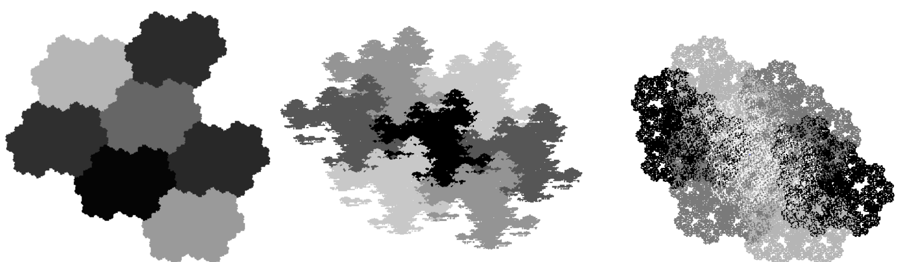
As for the Tribonacci substitution, there is no problem to factorize the Rauzy fractal through a lattice on an compact abelian group, so that the exchange of domains reduces to a group translation.

The question is the same as before: is this representation one-to-one? Unfortunately, the methods used for the Tribonacci substitution are quite specific and cannot be generalized. Anyway, some researches on that direction allow to deduce from the factorization of Rauzy fractal on compact abelian groups some combinatorial conditions for pure discrete spectrum. These conditions are based either on graphs [Sie04, Thu] or on the notion of balanced pairs [BK04, RI]. The problem is that the conditions are not general and need to be checked by hand on each substitution.

Since each example of a substitution of Pisot type that have been tested has a pure discrete spectrum, the point now is to exhibit some families of substitutions that provide a pure discrete spectrum dynamical system.

5. Conclusion

As a conclusion, we would like to emphasize the fact that the results exposed here mainly deal with spectral theory but can be also be expressed in more geometrical terms. Indeed, pure discrete spectrum has a nice geometrical equivalent in the unimodular case: thanks to the geometrical representation with Rauzy fractal, it is proved that a substitution of Pisot type with coincidence has a pure discrete spectrum if and only if its Rauzy fractals generates a periodic tiling of the plane [Sie04, RI, BK04]. Hence, conditions for pure discrete spectrum discussed above allows one to prove that the Rauzy fractals generated by the Tribonacci substitution, the substitution $1 \mapsto 11223$, $2 \mapsto 123$, $3 \mapsto 2$, or the substitution $1 \mapsto 12$, $2 \mapsto 3$, $3 \mapsto 1$ generate a periodic tiling. More generally, all the Rauzy fractal showed before do generate a periodic tiling.



Tribonacci substitution | $1 \mapsto 11223$, $2 \mapsto 123$, $3 \mapsto 2$ | $1 \mapsto 12$, $2 \mapsto 3$, $3 \mapsto 1$

Figure 7. Periodic tilings generated by Rauzy fractals.

Hence, substitutions have relations with a quite large number of mathematical domains (further illustrations are given in [PF02]). Combination of combinatorics, spectral theory, geometry and number theory will allow now to consider and apply this simple combinatorial object (a substitution) in different directions:

- proving general results on discrete spectrum and tilings;
- application to β-numeration and diophantine analysis [Bas02, Aki99];
- Generation of discrete planes [ABI02, ABS04];
- Models for quasi-crystals [Sen95, Lag99];
- Construction of explicit Markov partitions for toral automorphisms [IO93, KV98].

References

[ABI02] P. Arnoux, V. Berthé, and S. Ito. Discrete planes, $\mathbb{Z}^2$-actions, Jacobi-Perron algorithm and substitutions. *Ann. Inst. Fourier (Grenoble)*, 52(2):305–349, 2002.

[ABS04] P. Arnoux, V. Berthé, and A. Siegel. Two-dimensional iterated morphisms and discrete planes. *Theor. Comp. Sci.*, 319:145–176, 2004.

[AFH99] P. Arnoux, S. Ferenczi, and P. Hubert. Trajectories of rotations. *Acta Arith.*, 87(3):209–217, 1999.

[AI01] P. Arnoux and S. Ito. Pisot substitutions and Rauzy fractals. *Bull. Belg. Math. Soc. Simon Stevin*, 8(2):181–207, 2001. Journées Montoises (Marne-la-Vallée, 2000).

[Aki99] S. Akiyama. Self affine tiling and Pisot numeration system. In *Number theory and its applications (Kyoto, 1997)*, volume 2 of *Dev. Math.*, pages 7–17. Kluwer Acad. Publ., Dordrecht, 1999.

[Bas02] F. Bassino. Beta-expansions for cubic Pisot numbers. In *LATIN 2002: Theoretical informatics (Cancun)*, volume 2286 of *Lecture Notes in Comput. Sci.*, pages 141–152. Springer, Berlin, 2002.

[BD02] M. Barge and B. Diamond. Coincidence for substitutions of Pisot type. *Bull. Soc. Math. France*, 130:619–626, 2002.

[BK04] M. Barge and J. Kwapisz. Geometric theory of unimodular pisot substitutions. preprint, 2004.

[Dek78] F. M. Dekking. The spectrum of dynamical systems arising from substitutions of constant length. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 41(3):221–239, 1977/78.

[dJ77] Andrés del Junco. A transformation with simple spectrum which is not rank one. *Canad. J. Math.*, 29(3):655–663, 1977.

[Dur98] F. Durand. A characterization of substitutive sequences using return words. *Discrete Math.*, 179(1-3):89–101, 1998.

[FMN96] S. Ferenczi, C. Mauduit, and A. Nogueira. Substitution dynamical systems: algebraic characterization of eigenvalues. *Ann. Sci. École Norm. Sup.*, 29(4):519–533, 1996.

[Gou97] F. Q. Gouvêa. *p-adic numbers*. Universitext. Springer-Verlag, Berlin, second edition, 1997. An introduction.

[Hos86] B. Host. Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable. *Ergodic Theory Dynam. Systems*, 6(4):529–540, 1986.

[HS03] M. Hollander and B. Solomyak. Two-symbol Pisot substitutions have pure discrete spectrum. *Ergodic Theory Dynam. Systems*, 23:533–540, 2003.

[IO93] S. Ito and M. Ohtsuki. Modified Jacobi-Perron algorithm and generating Markov partitions for special hyperbolic toral automorphisms. *Tokyo J. Math.*, 16(2):441–472, 1993.

[Kam70] T. Kamae. Spectrum of a substitution minimal set. *J. Math. Soc. Japan*, 22:567–578, 1970.

[KV98] R. Kenyon and A. Vershik. Arithmetic construction of sofic partitions of hyperbolic toral automorphisms. *Ergodic Theory Dynam. Systems*, 18(2):357–372, 1998.

[Lag99] J. C. Lagarias. Geometric models for quasicrystals I. Delone sets of finite type. *Discrete Comput. Geom.*, 21(2):161–191, 1999.

[Liv87] A. N. Livshits. On the spectra of adic transformations of Markov compact sets. *Uspekhi Mat. Nauk*, 42(3(255)):189–190, 1987. English translation: *Russian Math. Surveys* 42(3): 222–223, 1987.

[LM95] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.

[Mar71] J. C. Martin. Substitution minimal flows. *Amer. J. Math.*, 93:503–526, 1971.

[Mor21] H. M. Morse. Recurrent geodesics on a surface of negative curvature. *Trans. Amer. Math. Soc.*, 22(1):84–100, 1921.

[PF02] N. Pytheas-Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics*. Lectures Notes in Mathematics 1794, Springer-Verlag, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[Que87] M. Queffélec. *Substitution dynamical systems—spectral analysis*. Lecture Notes in Mathematics, 1294. Springer-Verlag, Berlin, 1987.

[Rau82] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[RI] H. Rao and S. Ito. On super-coincidence condition. preprint.

[Sen95] Marjorie Senechal. *Quasicrystals and geometry*. Cambridge University Press, Cambridge, 1995.

[Sie03] A. Siegel. Représentation des systèmes dynamiques substitutifs non unimodulaires. *Ergodic Theory Dynam. Systems*, 23:1247–1273, 2003.

[Sie04] A. Siegel. Pure discrete spectrum dynamical system and periodic tiling associated with a substitution. *Ann. Inst. Four.*, 2(54), 2004.

[Sol92] B. Solomyak. On the spectral theory of adic transformations. In *Representation theory and dynamical systems*, pages 217–230. Amer. Math. Soc., Providence, RI, 1992.

[Thu] J. Thuswaldner. Unimodular pisot substitutions and their associated tiles. preprint.

[VL92] A. M. Vershik and A. N. Livshits. Adic models of ergodic transformations, spectral theory, substitutions, and related topics. In *Representation theory and dynamical systems*, pages 185–204. Amer. Math. Soc., Providence, RI, 1992.

[Wal82] P. Walters. *An introduction to ergodic theory*. Springer-Verlag, New York, 1982.

IRISA CNRS-UMR 6074 - Campus de Beaulieu - 35042 Rennes Cedex - France
E-mail address: Anne.Siegel@irisa.fr