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Stochastic constructions of flows of rank 1

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Abstract. Automorphisms of rank 1 appeared in the well-known papers of Chacon (1965), who constructed an example of a weakly mixing automorphism not having the strong mixing property, and Ornstein (1970), who proved the existence of mixing automorphisms without a square root. Ornstein's construction is essentially stochastic, since its parameters are chosen in a 'sufficiently random manner' according to a certain random law. In the present article it is shown that mixing flows of rank 1 exist. The construction given is also stochastic and is based to a large extent on ideas in Ornstein's paper. At the same time it complements Ornstein's paper and makes it more transparent. The construction can be used also to obtain automorphisms with various approximation and statistical properties. It is established that the new examples of dynamical systems are not isomorphic to Ornstein automorphisms, that is, they are qualitatively new.

Bibliography: 19 titles.

The fundamental ideas in the theory of approximation of dynamical systems, which is one of the actively evolving areas of modern ergodic theory, were put forth at the end of the 1960s in publications of Katok, Oseledets, and Stepin (see [1]–[5]). Several years later they had been broadly developed and had become an integral part of many investigations in the area of dynamical systems, among which we mention the work of Chacon, Keane, Ornstein, Sataev, and others (see [6]–[9]). The approximation theory method consists in approximating a given action with invariant measure (a cascade, flow, and so on) by an action with a simple structure, for example, periodic. This method turns out to be very useful in the spectral theory of dynamical systems, for example, in questions involving *spectral multiplicity* (see [1], [9]) or the *group property of the spectrum*, introduced by Kolmogorov. Recently Ageev and Ryzhikov applied the approximation theory method to the solution of the Rokhlin problem on the existence of a transformation with homogeneous spectrum of multiplicity 2. Another illustration of the foregoing is the investigation of a number of metric properties of flows on surfaces of genus $p \geqslant 1$ (see [4], [5]) by reduction to the study of *interval exchange* transformations, systems having nice approximation properties ([7], [10]).

At the basis of the theory of approximation of dynamical systems is the well-known Rokhlin–Halmos lemma (see [11]) asserting that if an automorphism T of a Lebesgue space $(X, \mathcal{A}, \mu)$ is aperiodic, then for any $\varepsilon > 0$ and $h \in \mathbb{N}$ there is a $B \in \mathcal{A}$

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such that the sets $B, TB, \dots, T^{h-1}B$ are disjoint and $\mu(B \sqcup TB \sqcup \dots \sqcup T^{h-1}B)$ is greater than $1 - \varepsilon$. A set of the form $U = B \sqcup TB \sqcup \dots \sqcup T^{h-1}B$ is called a Rokhlin–Halmos tower of height h.

The study of approximation properties of dynamical systems can be reduced in a natural way to the investigation of systems of rank 1. An automorphism T is called a transformation of rank 1 if there is a sequence of partitions $\xi_n = \{C_n, TC_n, T^2C_n, \dots, T^{h-1}C_n, Y_n\}$ of X that converges to the partition into points (this means that any measurable set can be approximated arbitrarily well by ξ_n-measurable sets). Automorphisms of rank 1 were first considered by Chacon [6] and Ornstein [12]. Actions of rank 1 include the set of classical dynamical systems such as systems with pure point spectrum (see [13]). Automorphisms of rank 1 also lie at the basis of many constructions, in particular, counterexamples in ergodic theory. For example, it is known that mixing transformations of rank 1 commute only with their powers T^k and as a consequence do not have roots.

In the theory of joinings, systems of rank 1 and their generalizations—systems of finite and local rank—make up one of the most thoroughly studied classes of actions. A *joining* (of second order) of an action $\{T^g\}_{g\in G}$ of a group G on a space $(X, \mathcal{A}, \mu)$ is defined to be a measure ν on the direct product $X \times X$ that is invariant with respect to the transformations $T^g \times T^g$, $g \in G$, and has the following property: the projections ν onto the direct factors coincide with μ. In contemporary ergodic theory the concept of a joining is one of the most effective tools for investigating dynamical systems (see [14] and also [15] and [16]).

For example, joinings arise naturally in investigations connected with the famous Rokhlin problem on multiple mixing [11]. An automorphism T is said to be *mixing with multiplicity* r if

$$\mu(T^{t_0}A_0 \cap T^{t_1}A_1 \cap \dots \cap T^{t_r}A_r) \to \mu(A_0)\mu(A_1)\cdots\mu(A_r)$$

as $|t_i - t_j| \to \infty$. The Rokhlin problem can be stated as follows: does simple mixing imply mixing of all orders? It has been proved that for automorphisms of finite rank the answer to Rokhlin's question is affirmative (see [17]). Moreover, for an automorphism of rank 1 the mixing property implies the property of *minimality of joinings* of second order, which consists in the set of all such joinings being exhausted by the measures $\mu \times \mu$ and $(\mathbb{I} \times T^k)\Delta$, where Δ is the diagonal joining

$$\Delta(A \times B) = \mu(A \cap B).$$

A generalization of this concept is the property of primeness (introduced by Veech, Rudolph, and del Junco; see [18] and [19]), which differs in that the existence of joinings of the form $(\mathbb{I} \times S)\Delta$ is allowed, where S is an arbitrary automorphism commuting with T.

In the present article we present a construction of mixing flows of rank 1. We thereby establish the existence of such flows and give new examples of prime flows.

Our construction is based on a universal construction of flows of rank 1, defined in detail in § 1. This construction is given by the following parameters: a sequence of heights h_n of approximating towers and a set of positive numbers $s_{n,j}$, $0 \leqslant j \leqslant q_n$. Our construction of mixing flows of rank 1 is stochastic in the following sense.

The parameters $s_{n,j}$ are chosen as realizations of independent random variables ξ_n (this is made more precise in § 2). Thus, a random family $\{T^t_\omega\}$ (ensemble) of flows is constructed, where ω runs through a corresponding probability space Ω (which is a Lebesgue space). It is established that under certain restrictions on the sequences h_n and $s_{n,j}$ the flow $\{T^t_\omega\}$ is almost surely mixing.

§ 1. A universal construction of a flow of rank 1

Let $\mathbf{T} = (T^t)_{t\in\mathbb{R}}$ be a measurable flow on a Lebesgue space $(X, \mathcal{A}, \mu)$, that is, a family of automorphisms T^t of X satisfying the condition $T^{t+u} = T^t T^u$. The flow $\mathbf{T}$ is said to be *measurable* if the function $F(t, x) := T^t x$ is measurable as a function of the two variables x and t.

Definition 1.1. Let $h > 0$ and let $\mathcal{B}[0, h]$ denote the σ-algebra of Borel subsets of $[0, h]$. A *Rokhlin–Halmos tower* (or simply a *tower*) of height h is defined to be a map $\varphi\colon U \to [0, h]$, $U \in \mathcal{A}$, with the following properties:

(a) the map φ is measurable, that is, $B \in \mathcal{B}[0, h] \Rightarrow \varphi^{-1}B \in \mathcal{A}$;

(b) if $B \in \mathcal{B}[0, h]$ and $S^t B \subseteq [0, h]$, then $T^t\varphi^{-1}B = \varphi^{-1}S^t B$, where $S^t x := x+t$.

A *tower* is sometimes defined to be a set $[\varphi] := U = \varphi^{-1}[0, h]$. Associated with each tower φ is the σ-algebra $\mathcal{A}(\varphi) \subseteq \mathcal{A}$ generated by the sets $\varphi^{-1}B$, $B \in \mathcal{B}[0, h]$, and $X \setminus [\varphi]$.

Definition 1.2. A flow $\mathbf{T}$ is said to be a *flow of rank* 1 if there is a sequence of towers φ_n with the following properties:

(a) $\mu[\varphi_n] \to 1$;

(b) $\mathcal{A}(\varphi_n) \to \mathcal{A}$, that is, for any set $A \in \mathcal{A}$ there exist sets $A_n \in \mathcal{A}(\varphi_n)$ such that $\mu(A \triangle A_n) \to 0$ as $n \to \infty$.

Remark 1.3. The approximating sequence of towers φ_n for a flow of rank 1 can be chosen so that $\mathcal{A}(\varphi_n) \subset \mathcal{A}(\varphi_{n+1})$.

This simple fact gives us that any flow of rank 1 can be obtained as a result of the following construction.

Universal construction of a flow of rank 1. Let $\{h_n\}_{n=0}^\infty$ be an increasing sequence of positive numbers, and let I_n be the circle $\mathbb{R}/h_n\mathbb{Z}$, identified with the closed interval $[0, h_n]$ with the end-points glued together. Suppose that $J_n \subset (0, h_{n+1})$ are given sets that are disjoint unions of open intervals $J_{n,k} = l_{n,k}+(0, h_n)$, $1 \leqslant k \leqslant q_n$. Let $\overleftarrow{\varphi}_n\colon I_{n+1} \to I_n$ be the natural projection acting according to the rule

$$\overleftarrow{\varphi}_n\colon l_{n,k} + t \mapsto t \quad \forall\, k, \forall\, t \in (0, h_n),$$
$$\overleftarrow{\varphi}_n\colon t \mapsto 0 \quad \forall\, t \notin J_n.$$

We remark that $\overleftarrow{\varphi}_n$ is a monotone continuous map from I_{n+1} to I_n of degree q_n. Let $\varphi_{j,n} = \overleftarrow{\varphi}_n \circ \dots \circ \overleftarrow{\varphi}_{j-1}$, $j > n$. Let $M_0 = h_0$ and define

$$M_{n+1} = \frac{\lambda(I_{n+1})}{\lambda(J_n)} M_n, \quad \text{where } \lambda \text{ is Lebesgue measure on } \mathbb{R}.$$

We assume that

$$M := \lim_{n\to\infty} M_n < \infty,$$

and we set $m_n := M_n/M$. On the circle I_n we define the probability measure $\mu_n = (1-m_n)\boldsymbol{\delta}_0 + m_n\lambda_n$, where $\boldsymbol{\delta}_0$ is the δ-function with support at zero and $\lambda_n := h_n^{-1}\lambda$ is normalized Lebesgue measure on I_n. Then $\varphi_{j,n}\mu_j = \mu_n$. We consider the set X of sequences $x = (t_0, t_1, \dots)$ such that $\overleftarrow{\varphi}_n t_{n+1} = t_n$. Let $\varphi_n \colon X \to I_n$ be the natural projection, $\varphi_n \colon (t_1, t_2, \dots) \mapsto t_n$. On the set X we consider the family of σ-algebras

$$\mathcal{A}_n := \{\varphi_n^{-1}B : B \in \mathcal{B}(I_n)\}, \qquad \mathcal{A}_n \subseteq \mathcal{A}_{n+1}, \quad \text{and let} \quad \mathcal{A} := \bigvee_n \mathcal{A}_n.$$

The measures μ_n can be regarded as measures on the σ-algebras $\mathcal{A}_n$. More precisely, let $\varphi_n^*\mu_n(\varphi_n^{-1}(B)) := \mu_n(B)$, where $B \in \mathcal{B}[0, h_n]$. The measures $\varphi_n^*\mu_n$ are consistent in the following sense: $\varphi_j^*\mu_j|_{\mathcal{A}_n} = \varphi_n^*\mu_n$ if $j \geqslant n$. Therefore, there is a unique measure μ defined on the σ-algebra $\mathcal{A}$ and such that $\mu|_{\mathcal{A}_n} = \varphi_n^*\mu_n$, and the space $(X, \mathcal{A}, \mu)$ constructed is a Lebesgue space. This space is called the *inverse limit of the spaces* $(I_n, \mathcal{B}(I_n), \mu_n)$.

We define the flow $\mathbf{T} = (T^t)$ on the space X as follows. Let $x = (t_1, \dots, t_n, \dots)$ be a point in X, and consider the sequence $(u_1, \dots, u_n, \dots)$, where $u_n := t_1 + t \pmod{h_n}$. It is not hard to see that for μ-almost all points x there is an n_0 such that $\overleftarrow{\varphi}_n u_{n+1} = u_n$ for all $n > n_0$. Let $T^t x$ be the sequence $(\widetilde{u}_1, \dots, \widetilde{u}_n, \dots)$ with $\widetilde{u}_n := u_n$ for $n \geqslant n_0$ and $\widetilde{u}_n := \varphi_{n_0,n} u_n$ for $n < n_0$. It is easy to show that T^t is a map preserving the measure μ and that $T^t T^u x = T^{t+u} x$ for μ-almost all x. Moreover, $\mathbf{T} = (T^t)$ is a flow of rank 1 with sequence of approximating towers $\varphi_n|_{U_n}$, where $U_n := \varphi_n^{-1}(0, h_n) = X \setminus \varphi_n^{-1}\{0\}$ and $\varphi_n|_{U_n}$ is the restriction of the function φ_n to the set U_n.

§ 2. The stochastic construction

The goal of this section is to describe in detail a stochastic construction of flows of rank 1 and to state the main theorem.

Namely, we construct a family $\{\mathbf{T}_\omega\}$ of flows of rank 1 parametrized by points ω in a probability space Ω for which it will be proved that, under certain conditions on the parameters of the construction, the flow $\mathbf{T}_\omega$ is almost surely mixing.

In § 1 we discussed the general construction of a flow of rank 1, in which the basic role is played by the maps $\overleftarrow{\varphi}_n \colon I_{n+1} \to I_n$ determined by the parameters h_n and $l_{n,k}$, $1 \leqslant k \leqslant q_n$. Noting that $l_{n,k+1} - l_{n,k} \geqslant h_n$, we define the new parameters $s_{n,k} = l_{n,k+1} - l_{n,k} - h_n$, and we let $s_{n,0} = l_{n,1}$ and $s_{n,q_n} = h_{n+1} - l_{n,q_n} - h_n$. The idea of the stochastic construction consists in regarding the parameters $s_{n,k}$ as independent realizations of a random variable ξ_n. This idea is made more precise as follows.

We fix a sequence of distributions ξ_n on $\mathbb{R}_+ = [0, \infty)$ such that $\xi_n \ll h_n$, that is, $\xi_n \leqslant \rho_n h_n$, where $\rho_n \to 0$ as $n \to \infty$.

Lemma 2.1. a) *There is a stationary random process Φ_n with continuous time that satisfies the following conditions*:

(1) $\Phi_n \colon \mathbb{R} \to I_n$ *is a continuous map*;

(2) *if* $\Phi_n(u)=a\in(0,h_n)$ *and* $t\leqslant h_n-a$, *then* $\Phi_n(u+t)=a+t$;
(3) *if* $[t_1,t_2]$ *is a connected component of the set* $\Phi_n^{-1}(0)$, *then the length of this closed interval is distributed like* ξ_n, *that is*, $\mathsf{P}\{t_2\geqslant t\mid t_1\}=\mathsf{P}\{\xi_n\geqslant t-t_1\}$.

b) *There is a Markov process* $\widetilde{\Phi}_n$ *such that*:

$(\widetilde{1})$ $\widetilde{\Phi}_n\colon\mathbb{R}\to I_n\sqcup\mathbb{R}^2_+$;
$(\widetilde{2})$ $\widetilde{\Phi}_n(t)=\Phi_n(t)$ *if and only if* $\Phi_n(t)\neq 0$, *and* $\widetilde{\Phi}_n(t)=(l^-,l^+)$ *if and only if* $\Phi_n(t)=0$ *and* $[t-l^-,t+l^+]$ *is the connected component of* $\Phi_n^{-1}(0)$ *containing* t.

Remark 2.2. The processes Φ_n and $\widetilde{\Phi}_n$ are naturally isomorphic: a realization of one of these processes can be uniquely recovered from a realization of the other. The definition of a random flow of rank 1 will involve the process Φ_n and the proofs will use the process $\widetilde{\Phi}_n$.

Remark 2.3. The processes Φ_n and $\widetilde{\Phi}_n$ exist if and only if the random variable ξ_n has finite variance. We mention also the important circumstance that the probability space of realizations of the process Φ_n is a Lebesgue space.

Proof of Lemma 2.1. Thus, we construct the auxiliary Markov process $\widetilde{\Phi}_n\colon\mathbb{R}\to I_n\sqcup\mathbb{R}^2_+$. For this it is convenient to introduce another Markov process Ψ equivalent to $\widetilde{\Phi}_n$. Namely, let the line $\mathbb{R}$ be partitioned into the half-open intervals $[t_k,t_{k+1})$ by the points t_k—the left-hand end-points of the open intervals of length h_n making up $\widetilde{\Phi}_n^{-1}(0,h_n)$—numbered arbitrarily. The processes $\widetilde{\Phi}_n$ and $\Psi\colon\mathbb{R}\to\mathbb{R}^2_+$ are connected by the following relation: $\Psi(t)=(l^-,l^+)$ if $t\in[t_k,t_{k+1})$, where $l^-=t-t_k$ and $l^+=t_{k+1}-t$. Let

$$P(s):=\begin{cases} p_n(s-h_n) & \text{if } s\geqslant h_n,\\ 0 & \text{otherwise,}\end{cases}$$

where p_n is the density of ξ_n. It is clear that P is the density of the random variable $h_n+\xi_n$. We consider the function $\widetilde{P}(l^\pm)=L^{-1}p_n(l^-+l^+)$, where $L=h_n+\overline{\xi}_n$, and we show that it is a stationary density for the Markov process Ψ, which we specify by the transition probabilities

$$\tau_\varepsilon(l^\pm,l^\pm_\varepsilon)=\begin{cases} \boldsymbol{\delta}_{l^\pm+(\varepsilon,-\varepsilon)}(l^\pm_\varepsilon) & \text{if } l^+>\varepsilon,\\ \boldsymbol{\delta}_{\varepsilon-l^+}(l^-_\varepsilon)P(l^-_\varepsilon+l^+_\varepsilon) & \text{if } l^+\leqslant\varepsilon,\end{cases}$$

where $0<\varepsilon<h_n$. The transition probabilities τ_u are defined as follows for arbitrary $u>0$. We consider some partition of $[0,u)$ into half-open intervals $[u_{k-1},u_k)$, $k=1,\dots,K$, such that $u_k-u_{k-1}<h_n$, and we set

$$\tau_u(l^\pm,B):=\int_{\mathbb{R}^2_+}\tau_{u_1}(l^\pm,dl^\pm_1)\int_{\mathbb{R}^2_+}\tau_{u_2-u_1}(l^\pm_1,dl^\pm_2)\cdots\int_B\tau_{u_K-u_{K-1}}(l^\pm_{K-1},dl_k),$$

where B is a measurable subset of $\mathbb{R}^2_+$. This definition can be shown to be unambiguous, that is, the measures defined do not depend on the choice of partition, and we do not dwell on this simple computation. We merely remark that it suffices to verify the correctness of the definition for $u \in (0, h_n)$, since from any pair of partitions we can pass to their least upper bound, while for small u the correctness (which is equivalent to the Chapman–Kolmogorov conditions) can be verified directly.

For $u < 0$ the measures τ_u are defined similarly.

We verify that the density $\widetilde{P}$ really is stationary. Let

$$\eta_\varepsilon(B) := \int_{\mathbb{R}^2_+} \widetilde{P}(l^\pm) \tau_\varepsilon(l^\pm, B)\, dl^\pm.$$

For a measurable set $B \subset \{l^-_\varepsilon > \varepsilon\}$ we have

$$\eta_\varepsilon(B) = \int_{(-\varepsilon,\varepsilon)+B} \widetilde{P}(l^\pm)\, dl^\pm = \int_B \widetilde{P}(l^\pm)\, dl^\pm,$$

and for $B \subset \{l^-_\varepsilon \leqslant \varepsilon\}$

$$\begin{aligned} &\int_{\{l^+ \leqslant \varepsilon\}} \widetilde{P}(l^\pm)\, dl^\pm \int_{\{l^+_\varepsilon : (\varepsilon - l^+, l^+_\varepsilon) \in B\}} P(\varepsilon - l^+ + l^+_\varepsilon)\, dl^+_\varepsilon \\ &\quad = \int_0^\varepsilon L^{-1}\, dl^+ \int_{\{l^+_\varepsilon : (\varepsilon - l^+, l^+_\varepsilon) \in B\}} P(\varepsilon - l^+ + l^+_\varepsilon)\, dl^+_\varepsilon \\ &\quad = L^{-1} \int_B P(z, l^+_\varepsilon)\, dz\, dl^+_\varepsilon = \int_B \widetilde{P}(l^\pm_\varepsilon)\, dl^\pm_\varepsilon. \end{aligned}$$

Thus, it is proved that the distribution determined by the density $\widetilde{P}$ is stationary.

We recall that there is a one-to-one correspondence between the processes Ψ and $\widetilde{\Phi}_n$. Let $\overline{m}_n := h_n/(h_n + \overline{\xi}_n)$. We compute the stationary density $\widetilde{p}_n$ for the process $\widetilde{\Phi}_n$, that is, the density corresponding to $\widetilde{P}$ of the distribution of the random variable $\widetilde{\Phi}_n(0)$:

$$\begin{aligned} \widetilde{p}_n(x) &= \int_0^\infty \widetilde{P}(x, l^-)\, dl^- = L^{-1} = h_n^{-1} \overline{m}_n, \qquad x \in I_n; \\ \widetilde{p}_n(l^\pm) &= \widetilde{P}(l^- + h_n, l^+) = h_n^{-1} \overline{m}_n\, p_n(l^- + l^+), \qquad l^\pm \in \mathbb{R}^2_+. \end{aligned}$$

To get the process Φ_n from the process $\widetilde{\Phi}_n$, it suffices to set $\Phi_n(t) = \widetilde{\Phi}_n(t)$ if $\Phi_n(t) \in I_n$, and $\Phi_n(t) = 0$ otherwise.

The properties (2), (3), and $(\widetilde{2})$ follow immediately from the definition of the transition probabilities τ_u.

We now have all the necessary data to proceed to the description of the stochastic construction of flows of rank 1. Let Ω be the probability space that is the direct product of the spaces of realizations of the random processes Φ_n; then the Φ_n are jointly independent. For each n we consider a realization of the random process $\Phi_n(t)$ and define the map $\overleftarrow{\varphi}_n$ as follows. If the interval $(t_0, t_0 + h_n)$ forming the set $\Phi_n^{-1}(I_n \setminus \{0\})$ is entirely contained in the interval $(0, h_{n+1})$, then we set $\overleftarrow{\varphi}_n(t) := \Phi_n(t)$ for $t \in (t_0, t_0 + h_n)$; otherwise, $\overleftarrow{\varphi}_n(t) := 0$.

We let $\sigma_n := \sqrt{\mathsf{D}\xi_n}$ and note that, since $\xi_n \geqslant 0$, it follows from the Cauchy–Bunyakovskiĭ–Schwarz inequality that

$$\overline{\xi}_n := \mathsf{E}\xi_n \leqslant \sigma_n.$$

Lemma 2.4. *If* $\overline{q}_n := h_{n+1}/h_n \to 1$ *and* $\sum_{n=1}^{\infty} \sigma_n/h_n < \infty$, *then the sequence* $\overleftarrow{\varphi}_n$ *almost surely correctly determines a space* X_ω *with a finite measure and a flow* $\mathbf{T}_\omega$ *of rank* 1 *on it.*

Let $\kappa_n := h_n^2/\sigma_n^2$. Suppose also that the random variable ξ_n has a density p_n that is a function of bounded variation. Let $\widehat{p}_n$ be the Fourier transform of the function p_n:

$$\widehat{p}_n(x) := \int_0^\infty e^{-2\pi i x t} p_n(t)\, dt.$$

Theorem 2.5. *Suppose that the flow* $\mathbf{T}_\omega$ *is the result of the above construction depending on the parameters* h_n *and* ξ_n, *and assume that the following conditions hold for some constants* $0 < \chi < 1$ *and* $\gamma, \epsilon, C_1, C_2 > 0$:

(1) $\|(1 + C_1\sigma_n^2 x^2)^{1/2}\, \widehat{p}_n(x)\|_\infty \leqslant 1$;
(2) $\prod_{j<n} \overline{q}_j \leqslant C_2\, \overline{q}_n^{\gamma}$;
(3) $\|\overline{q}_{n-1}^{-\chi} h_n p_n\|_\infty \to 0$;
(4) $\dfrac{\kappa_n}{\overline{q}_n^{1-\chi-\epsilon}} \to 0$.

Then the flow $\mathbf{T}_\omega$ *is mixing with probability* 1.

§ 3. Example of a family of mixing flows of rank 1

We now present a corollary to Theorem 2.5 giving a simple yet sufficiently general example of an ensemble of mixing flows of rank 1. Let us fix a distribution ξ on $\mathbb{R}_+$ with density $p(t)$ such that $\xi \leqslant M < \infty$. Let ξ_n have the density $p_n := p(t/\sigma_n)/\sigma_n$. In this case Theorem 2.5 takes the following form.

Corollary 3.1. *If*

$$\overline{q}_{n-1}^{-\chi}\sqrt{\kappa_n} \to 0, \qquad \frac{\kappa_n}{\overline{q}_n^{1-\chi-\epsilon}} \to 0, \qquad \text{and} \qquad \prod_{j<n} \overline{q}_j \leqslant C_2 \overline{q}_n^{\gamma},$$

then the flow $\mathbf{T}_\omega$ *is mixing with probability* 1.

Proof. We verify the conditions (1) of the theorem. Indeed, since the variance of the random variable ξ is finite, there is a constant C_1 such that

$$\widehat{p}(t) \leqslant (1 + C_1 x^2)^{-1/2}, \qquad t \in \mathbb{R}.$$

But $p_n(x) = p(\sigma_n x)$, and hence

$$\|(1 + C_1\sigma_n^2 x^2)^{1/2}\widehat{p}_n(x)\| \leqslant 1.$$

The condition (2) in Theorem 2.5 takes the form indicated in the corollary, because $\|p_n\|_\infty = \sigma_n^{-1}\|p\|_\infty$. The corollary is proved.

Let us now consider a more concrete example. Let $h_n = 2^{m^n}$ and $\sigma_n = 2^{\alpha m^n}$. Then $\overline{q}_n = 2^{(m-1)m^n}$ and $\kappa_n = 2^{2(1-\alpha)m^n}$. The fourth condition in the theorem follows from the estimate

$$\prod_{j<n} \overline{q}_j = 2^{(m-1)\sum_{j<n} m^n} \leqslant \overline{q}_n^{1/(m-1)},$$

and the conditions (2) and (3) take the form

$$\frac{\overline{q}_{n-1}^{-\chi} h_n}{\sigma_n} \to 0, \qquad \frac{\overline{q}_n^{1-\chi-\epsilon}}{\kappa_n} \to \infty,$$

that is,

$$-\frac{\chi(m-1)}{m} + 1 - \alpha < 0, \qquad (1-\chi)(m-1) - 2(1-\alpha) > 0;$$

hence

$$\frac{m(1-\alpha)}{m-1} < \chi < \frac{2\alpha + m - 3}{m-1}\,.$$

Obviously, the desired constant χ exists if and only if $m(1-\alpha) < 2\alpha + m - 3$, that is,

$$\alpha > \frac{3}{2+m}\,.$$

§ 4. Scheme of proof of the main theorem

The proof of Theorem 2.5 is constructed as follows. The direct proof, which makes up the content of § 7, is preceded by the necessary probabilistic constructions, the formulation of which is independent of the main result, and by the proofs of all the necessary lemmas. Namely, in § 5 we establish some theorems on large deviations for a special type of random processes with continuous time, and then in § 6 we investigate the asymptotic behaviour of the random processes Φ_n used in the stochastic construction. Finally, in § 7 we prove the main theorem on the basis of the results obtained.

§ 5. Lemmas on large deviations

A *discrete random process* is defined to be a two-sided sequence of random variables η_j with values in some finite alphabet $\mathbb{A}$. A process is said to be *stationary* if the joint distribution μ_k of the variables $\eta_{s+1}, \dots, \eta_{s+k}$ does not depend on s. The measure μ_k is given on the finite set of *k-blocks*: the sequences (words) $a_1^k = a_1 a_2 \dots a_k$ with $a_j \in \mathbb{A}$. Let μ be the measure on $\mathbb{A}^\infty := \prod_{j\in\mathbb{Z}} \mathbb{A}$ corresponding to the process η.

Let us consider an n-block $x_1^n = x_1 x_2 \dots x_n$. We use the expression x_i^j to denote the subword $x_i x_{i+1} \dots x_j$. The *empirical distribution* of order k with respect to x_1^n is defined to be the measure $p_k(\,\cdot \mid x_1^n)$ on $\mathbb{A}^k$ defined by

$$p_k(a_1^k \mid x_1^n) = \frac{N_k(a_1^k)}{n-k+1}\,, \qquad \text{where} \quad N_k(a_1^k) := \#\{j : x_j^{j+k-1} = a_1^k\}.$$

Lemma 5.1. *Assume that the random variables η_j are independent. Then there is an absolute constant c_0 such that*

$$\mu\{x : \|p_1(\,\cdot \mid x_1^n) - \mu_1\| > \varepsilon\} \leqslant (n+1)^{\#\mathbb{A}} e^{-c_0 n \varepsilon^2} \qquad \forall\, \varepsilon > 0,$$

where $\|\cdot\|$ denotes the variation norm in the space of measures.

We fix an $s \in [0, k+g)$, where $g \leqslant k < n$ are positive integers, and we consider the (random) sequence of k-blocks

$$w_t^{(s)} := x_{s+t(k+g)}^{s+t(k+g)+k-1}, \qquad 0 \leqslant t < \mathcal{T}^{(s)},$$

where $\mathcal{T}^{(s)}$ is the total number of blocks $w_t^{(s)}$ in x_1^n. We define

$$N^{(s)}(a_1^k) := \#\{t : w_t^{(s)} = a_1^k\} \quad \text{and} \quad p_k^{(s)}(a_1^k) := \frac{N^{(s)}(a_1^k)}{\mathcal{T}^{(s)}}\,.$$

Lemma 5.2. *If $n > 8k$ and $g \leqslant k$, then*

$$\left| p_k(a_1^k \mid x_1^n) - \frac{1}{k+g} \sum_{s=0}^{k+g-1} p_k^{(s)}(a_1^k \mid x_1^n) \right| \leqslant \frac{10k}{n} p_k(a_1^k \mid x_1^n).$$

Proof. Obviously,

$$N(a_1^k) = \sum_{s=0}^{k+g-1} N^{(s)}(a_1^k) \quad \text{and} \quad n - 2(k+g) \leqslant (k+g)\mathcal{T}^{(s)} \leqslant n.$$

Using these relations, we get that

$$\begin{aligned}
&\left| p_k(a_1^k \mid x_1^n) - \frac{1}{k+g} \sum_{s=0}^{k+g-1} p_k^{(s)}(a_1^k \mid x_1^n) \right| \\
&\quad = \left| \frac{N(a_1^k)}{n-k+1} - \frac{1}{k+g} \sum_{s=0}^{k+g-1} \frac{N^{(s)}(a_1^k)}{\mathcal{T}^{(s)}} \right| \\
&\quad \leqslant \sum_{s=0}^{k+g-1} N^{(s)}(a_1^k) \left| \frac{1}{n-k+1} - \frac{1}{(k+g)\mathcal{T}^{(s)}} \right| \\
&\quad \leqslant \frac{k+2(k+g)}{n-2(k+g)} \sum_{s=0}^{k+g-1} \frac{N^{(s)}(a_1^k)}{n-k+1} \\
&\quad \leqslant \frac{5k}{n-4k} p_k(a_1^k \mid x_1^n) \leqslant \frac{10k}{n} p_k(a_1^k \mid x_1^n),
\end{aligned}$$

since $4k < n/2$. This is what was to be proved.

Corollary 5.3. *If $A \subseteq \mathbb{A}^k$ and $n > 8k$, then*

$$\left| p_k(A \mid x_1^n) - \frac{1}{k+g} \sum_{s=0}^{k+g-1} p_k^{(s)}(A \mid x_1^n) \right| \leqslant \frac{10k}{n}\,.$$

Corollary 5.4. *Let $A \subseteq \mathbb{A}^k$ and $n > 8k$. Then the inequality $|p_1(A \mid x_1^n) - p_k(A \mid x_1^n)| \leqslant 2k/n$ implies the estimate*

$$\left| p_1(A \mid x_1^n) - \frac{1}{k+g} \sum_{s=0}^{k+g-1} p_k^{(s)}(A \mid x_1^n) \right| \leqslant \frac{12k}{n}.$$

Definition 5.5. For words (blocks) w' and w'' of respective lengths ℓ' and ℓ'' we define the event $w'\langle g\rangle w''$ to be the set of sequences x such that $x_1^{\ell'} = w'$ and $x_{\ell'+g+1}^{\ell'+g+\ell''} = w''$. We say that the process η_j is *ψ-mixing*, where $\psi\colon \mathbb{N} \to [1, +\infty]$, if

$$\mu(w'\langle g\rangle w'') \leqslant \psi(g)\mu(w')\mu(w'')$$

for any w' and w'' of the indicated form.

Lemma 5.6. *Let η be a ψ-mixing stationary process with the alphabet $\mathbb{A}$ and let $n > 24k/\varepsilon$ and $\varepsilon < 1$. Then*

$$\mu\{x : \|p_1(\,\cdot \mid x_1^n) - \mu_1\| > \varepsilon\#\mathbb{A}\} \leqslant \#\mathbb{A}\cdot 2k\left(\frac{n}{2k}+1\right)^2 \psi(k)^{n/(2k)} \exp\left(-\frac{c_0\varepsilon^2}{8}\,\frac{n}{2k}\right).$$

Proof. Let $A := \{x : x_1 = a\}$ and $g := k$. We assume that

$$|p_1(A \mid x_1^n) - \mu(A)| > \varepsilon;$$

then in view of Corollary 5.4 there is an s such that

$$|p_k^{(s)}(A \mid x_1^n) - \mu(A)| > \varepsilon - \frac{12k}{n} > \frac{\varepsilon}{2}.$$

We consider the Bernoulli random process $\widetilde{\varkappa}_t \in \{0,1\}$ given by the probability vector $(\mu(A), 1-\mu(A))$, and we compare it with the process

$$\varkappa_t = \begin{cases} 1 & \text{if } w_t^{(s)} \in A, \\ 0 & \text{otherwise,} \end{cases}$$

induced by the process η. By Lemma 5.1,

$$\gamma\{\widetilde{x} : \|\widetilde{p}_1(\,\cdot \mid \widetilde{x}_1^n) - \gamma_1\| > \varepsilon'\} \leqslant \mathcal{E} = (\mathcal{T}^{(s)}+1)^2 \exp(-c_0{\varepsilon'}^2\mathcal{T}^{(s)}) \qquad \forall\, \varepsilon' > 0,$$

where γ is the measure corresponding to the process $\widetilde{\varkappa}_t$, and $\widetilde{p}_k$ is the empirical distribution constructed from γ. Using the ψ-mixing property, we get that

$$\mu\left\{x : |p_k^{(s)}(A \mid x_1^n) - \mu(A)| > \frac{\varepsilon}{2}\right\} \leqslant \psi(k)^{\mathcal{T}^{(s)}} \gamma\left\{\widetilde{x} : \|\widetilde{p}_1(\,\cdot \mid \widetilde{x}_1^{\mathcal{T}^{(s)}}) - \gamma_1\| > \frac{\varepsilon}{2}\right\}$$
$$\leqslant \psi(k)^{\mathcal{T}^{(s)}} (\mathcal{T}^{(s)}+1)^2 \exp\left(-c_0\mathcal{T}^{(s)}\frac{\varepsilon^2}{4}\right).$$

Thus, the probability of the event of interest to us can be estimated by the quantity

$$\#\mathbb{A}\cdot 2k\left(\frac{n}{2k}+1\right)^2 \psi(k)^{n/(2k)} \exp\left(-\frac{c_0\varepsilon^2}{8}\,\frac{n}{2k}\right), \quad \text{since} \quad g+k = 2k \quad \text{and} \quad \mathcal{T}^{(s)} \geqslant \frac{n}{4k}.$$

Definition 5.7. Let N be fixed, and suppose that the stationary random process $\eta(t)$ satisfies the following weaker property than ψ-mixing:

$$\mu(w'\langle g\rangle w'') \leqslant \psi(g)\mu(w')\mu(w'')$$

if $w' = w'^{s'+l'}_{s'}$ and $w'' = w''^{s''+l''}_{s''}$ are blocks such that $s', s'', s'+l', s''+l''$ belong to $\{1,\dots,N\}$, and $\psi\colon \mathbb{N}\to[1,+\infty]$. We call this the ${}^{(N)}\psi$*-mixing property.*

We need the following modification of Lemma 5.6, which encompasses ${}^{(N)}\psi$-mixing processes.

Lemma 5.8. *Let η be an ${}^{(N)}\psi$-mixing stationary process with the alphabet $\mathbb{A}$, assume that the requirement stated above holds, and let $24k/\varepsilon < n < N$ and $\varepsilon < 1$. Then*

$$\mu\bigl\{x : \|p_1(\,\cdot\mid x_1^n) - \mu_1\| > \varepsilon\#\mathbb{A}\bigr\} \leqslant \#\mathbb{A}\cdot 2k\biggl(\frac{n}{2k}+1\biggr)^2\psi(k)^{n/(2k)}\exp\biggl(-\frac{c_0\varepsilon^2}{8}\,\frac{n}{2k}\biggr).$$

The proof repeats word-for-word that of Lemma 5.6.

Theorem 5.9. *Suppose that η is a ψ-mixing or ${}^{(N)}\psi$-mixing stationary process,*

$$\psi(k) = \begin{cases} 1+e^{c-\omega k} & \text{if } k\geqslant k_0,\\ \infty & \text{otherwise,}\end{cases}$$

with alphabet $\mathbb{A}=\{0,1\}$, and assume the conditions of Lemma 5.6 or 5.8, respectively. Further, assume that

$$k_0 \leqslant \frac{\ln n + c}{\omega} \leqslant n^{2/3}, \qquad \frac{\omega n}{\ln n + c} > 24\biggl(\frac{64}{c_0\varepsilon^2}\biggr)^2 > 24.$$

Then

$$\mu\{x : \|p_1(\,\cdot\mid x_1^n) - \mu_1\| > \varepsilon\} \leqslant C_1\exp\biggl(-c_1\varepsilon^2\frac{\omega n}{\ln n + c}\biggr).$$

Proof. The estimate of the probability of a large deviation has the form

$$\mu\bigl\{x : \|p_1(\,\cdot\mid x_1^n) - \mu_1\| > \varepsilon\bigr\} \leqslant 4k\biggl(\frac{n}{2k}+1\biggr)^2(1+e^{c-\omega k})^{n/(2k)}\exp\biggl(-\frac{c_0\varepsilon^2}{32}\,\frac{n}{2k}\biggr).$$

Setting $k := (\ln n + c)/\omega$, we have

$$\begin{aligned}\mu\{x : \|p_1(\,\cdot\mid x_1^n) - \mu_1\| > \varepsilon\} &\leqslant 4k\biggl(\frac{n}{2k}+1\biggr)^2\biggl(1+\frac{1}{n}\biggr)^{n/(2k)}\exp\biggl(-\frac{c_0\varepsilon^2}{32}\,\frac{n}{2k}\biggr)\\ &\lesssim 8\biggl(\frac{n}{2k}\biggr)^2\exp\biggl(-\frac{c_0\varepsilon^2}{32}\,\frac{n}{2k}\biggr) \overset{(\flat)}{\lesssim} \exp\biggl(-\frac{c_0\varepsilon^2}{64}\,\frac{n}{2k}\biggr) = \exp\biggl(-c_1\varepsilon^2\frac{\omega n}{\ln n + c}\biggr),\end{aligned}$$

which is what was required to prove. Here we have used the following chain of arguments. First, $(1-n^{-1})^n \nearrow e$ as $n\to\infty$, and $k>1$, therefore, the sequence $(1-n^{-1})^{n/(2k)}$ is uniformly bounded. Further, let $z = n/(2k)$. Then $z \geqslant 3\gamma^{-1}\ln(24\gamma^{-1})$, where $\gamma := c_0\varepsilon^2/64$, since by assumption $z \geqslant 12\gamma^{-2} = 3\gamma^{-1}\cdot 4\gamma^{-1}$ and $4\gamma^{-1} \geqslant \ln(24\gamma^{-2})$ for any $\gamma\leqslant 1$. The inequality obtained can be rewritten in the form $u \geqslant 3\ln(3m)$, where $u=\gamma z$ and $m := 8\gamma^{-2}$. The last inequality is easily seen to imply that $mu^2 \leqslant e^u$ or, which is the same, $8z^2 \leqslant e^{\gamma z}$, and this proves the inequality $(\flat)$.

We proceed to the consideration of a stationary random process $\eta(t)\in\mathbb{A}=\{0,1\}$ with continuous time.

Definition 5.10. We consider sets w' and w'' of realizations of the random process η that are measurable with respect to the coordinate σ-algebras corresponding to the closed intervals

$$[t', t'+l'] \quad \text{and} \quad [t'', t''+l''].$$

Assume that $t'' > t' + l'$. We say that the process η is ${}^{(N)}\psi$*-mixing*, where $\psi\colon \mathbb{R}_+ \to [1, +\infty]$, if for any such events w' and w''

$$\mu(w' \wedge w'') \leqslant \psi(t'' - (t' + l'))\mu(w')\mu(w'').$$

Accordingly, suppose that the process η is ${}^{(N)}\psi$-mixing with the function

$$\psi(t) = \begin{cases} 1 + e^{\mathbf{c} - \boldsymbol{\omega} t} & \text{if } t \geqslant t_0, \\ \infty & \text{otherwise.} \end{cases}$$

We assume that the set

$$\mathcal{I}_\eta = \{t \in \mathbb{R} : \eta(t) = 1\}$$

is (μ-almost surely) a union of closed intervals.

Theorem 5.11. *Suppose that for some constants* $\boldsymbol{\rho}$ *and* $\mathbf{h}$

$$\#(\partial \mathcal{I}_\eta \cap [0, t]) \leqslant \frac{\boldsymbol{\rho} t}{\mathbf{h}} \quad \textit{and} \quad \boldsymbol{\omega}\tau \leqslant \mathbf{c}, \quad \textit{where} \quad \tau := \frac{\varepsilon \mathbf{h}}{4\boldsymbol{\rho}},$$

and, moreover,

$$\frac{t_0}{\tau} \leqslant \frac{\ln(T/\tau) + \mathbf{c}}{\boldsymbol{\omega}\tau} \leqslant \left(\frac{T}{\tau}\right)^{2/3}, \qquad \frac{\boldsymbol{\omega} T}{\ln(T/\tau) + \mathbf{c}} > 24\left(\frac{2^8}{c_0 \varepsilon^2}\right)^2 > 24.$$

Then there is a constant c_3 *such that*

$$\mu\left\{\left|\frac{1}{T}\int_0^T \eta(t)\,dt - \overline{\eta}\right| > \varepsilon\right\} \leqslant \exp\left(-c_3 \varepsilon^2 \frac{\boldsymbol{\omega} T}{\ln T - \ln(\varepsilon \mathbf{h}/(4\boldsymbol{\rho})) + 2\mathbf{c}}\right).$$

Proof. Setting $\tau := \varepsilon \mathbf{h}/(4\boldsymbol{\rho})$, we consider the stationary sequence

$$\widetilde{\eta}_j = \begin{cases} 1 & \text{if } \eta(t) = 1 \quad \forall\, t \in [j\tau, (j+1)\tau], \\ 0 & \text{otherwise.} \end{cases}$$

And setting

$$\eta^{(\tau)}(t) := \int_0^\tau \eta(t+u)\,du,$$

we note that $\mathsf{E}\eta^{(\tau)} = \overline{\eta}$. Therefore, it follows from the stationary property that

$$|\mathsf{E}\widetilde{\eta} - \overline{\eta}| = |\mathsf{E}\widetilde{\eta} - \mathsf{E}\eta^{(\tau)}| \leqslant \mu\{\eta : \partial \mathcal{I}_\eta \cap [0, \tau] \neq \varnothing\} \leqslant \frac{\boldsymbol{\rho}\tau}{\mathbf{h}} = \frac{\varepsilon}{4}.$$

It follows from the first condition of the theorem that

$$\left|\frac{1}{T}\int_0^T \eta(t)\,dt - p_1(1\mid x_1^n)\right| \leqslant \frac{\#(\partial\mathfrak{I}_\eta\cap[0,T])}{T/\tau} \leqslant \frac{\boldsymbol{\rho}T/\mathbf{h}}{T/(\varepsilon\mathbf{h}/(4\boldsymbol{\rho}))} = \frac{\varepsilon}{4}\,.$$

Further, the process η_j is $\widetilde{\psi}$-mixing with the function

$$\widetilde{\psi}(j) = \psi((j-1)\tau).$$

By the preceding lemma,

$$\mu\biggl\{x : |p_1(1\mid x_1^n) - \mu_1(1)| > \frac{\varepsilon}{2}\biggr\} \leqslant \exp\biggl(-\frac{c_1\varepsilon^2}{4}\,\frac{\boldsymbol{\omega}T}{\ln T - \ln\tau + \boldsymbol{\omega}\tau + \mathbf{c}}\biggr).$$

Thus,

$$\begin{aligned}\left|\frac{1}{T}\int_0^T \eta(t)\,dt - \overline{\eta}\right| &\leqslant \left|\frac{1}{T}\int_0^T \eta(t)\,dt - p_1(1\mid x_1^n)\right| + |p_1(1\mid x_1^n) - \mathsf{E}\eta_j| + |\mathsf{E}\eta_j - \overline{\eta}| \\ &\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

It remains to let $c_2 := c_1/4$. The proof of the theorem is complete.

§ 6. Some limit theorems

In this section we investigate properties of the random process $\Phi_n(t)$. Namely, we establish a property of $\Phi_n(t)$ analogous to the exponential mixing property. Since Theorem 2.5 is close in methods of proof to assertions about convergence of convolutions of regular probability measures on the circle, we begin by considering this simpler case as an illustration. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Definition 6.1. A *Gaussian measure* on $\mathbb{T}$ is defined to be a measure of the form

$$\zeta(B) = \int_{B+\mathbb{Z}} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-a)^2/(2\sigma^2)}\,du, \qquad B\in\mathcal{B}(\mathbb{T}), \quad \mathbb{T}\simeq[0,1),$$

where a and σ are constants.

We consider the random variables

$$\zeta_n := \pi_n^*(\boldsymbol{\delta}_{-\overline{\xi}_n} * \xi_n)^{*\lfloor\kappa_n\rfloor},$$

where $\pi_n\colon I_n\to\mathbb{T}$ is the natural projection.

Lemma 6.2. *Suppose that the distributions ξ_n satisfy the conditions of Theorem 2.5. Then the densities of the measures ζ_n converge uniformly to the density $p_{\mathrm{g}}(u)$ of the Gaussian measure with parameters $(0,1)$, that is,*

$$\|\pi_n^*(\boldsymbol{\delta}_{-\overline{\xi}_n} * p_n)^{*\lfloor\kappa_n\rfloor} - p_{\mathrm{g}}\|_\infty \to 0 \quad \textit{as } n\to\infty,$$

where $\pi_n^*(\boldsymbol{\delta}_{-\overline{\xi}_n} * p_n)$ *is the density of the measure* $\pi_n^*(\boldsymbol{\delta}_{-\overline{\xi}_n} * \xi_n)$:

$$\pi_n^*(\boldsymbol{\delta}_{-\overline{\xi}_n} * p_n)(u) = h_n p_n(\overline{\xi}_n + u h_n).$$

We remark that $p_{\mathrm{g}}(u) > 1/2$, and thus it can be assumed without loss of generality that $|p_n^{*\lfloor\kappa_n\rfloor}(t)| > 1/2$ for any n, where the convolution is defined in the natural way

$$(p_1 * p_2)(t) := \int_{I_n} p_1(w) p_2(t-w)\, dw.$$

Proof of Lemma 6.2. We denote by $P_n(t) = h_n p_n(\overline{\xi}_n + t h_n)$, $t \in I_n$, the density of the measure $\pi_n^*(\boldsymbol{\delta}_{-\overline{\xi}_n} * \xi_n)$. To establish the uniform convergence of the densities of the measures ζ_n to the density $p_{\mathrm{g}}(u)$ of the Gaussian measure it suffices to prove convergence in $L^1(\mathbb{R})$ of the functions $\widehat{P}_n^{\lceil\kappa_n\rceil}$ to the function $e^{-2\pi^2 x^2}$, where $\widehat{P}_n(x)$ is the Fourier transform of the function $P_n(t)$:

$$\widehat{P}_n(x) := \int_{I_n} e^{-2\pi i x t} P_n(t)\, dt, \qquad x \in h_n^{-1}\mathbb{Z}.$$

We show first that $\widehat{P}_n^{\lfloor\kappa_n\rfloor}(x) \to e^{-2\pi^2 x^2}$ at any point x. Indeed,

$$\widehat{P}_n(x) = 1 - \frac{2\pi^2 x^2}{\kappa_n} + \mathcal{E}_n(x), \qquad \kappa_n = \frac{h_n^2}{\sigma_n^2}\,.$$

To estimate $\mathcal{E}_n(x)$, we assume for the time being that P_n is a function on $\mathbb{R}$ (in this case $\widehat{P}_n$ is also defined on $\mathbb{R}$, and not just at points of the form $k h_n^{-1}$, $k \in \mathbb{Z}$). Then

$$\mathcal{E}_n(x) = \sum_{r \geqslant 3} \frac{\widehat{P}_n^{(r)}(0)}{r!} x^r,$$

because $\widehat{P}_n$ is an entire function. Noting that

$$\widehat{P}_n^{(r)}(0) = (2\pi i)^r \int_0^{\rho_n} t^r P_n(t)\, dt, \qquad \operatorname{supp} \xi_n \subseteq [0, \rho_n h_n],$$

we have

$$|\widehat{P}_n^{(2)}(0)| \leqslant 4\pi^2 \int_0^{\rho_n} t^2 P_n(t)\, dt = \frac{4\pi^2}{\kappa_n} \Longrightarrow |\widehat{P}_n^{(r)}(0)| \leqslant \frac{(2\pi)^r \rho_n^{r-2}}{\kappa_n}\,,$$

from which it follows that

$$\begin{aligned} |\mathcal{E}_n(x)| &\leqslant \frac{1}{\kappa_n \rho_n^2} \sum_{r \geqslant 3} \frac{|2\pi\rho_n x|^r}{r!} = \frac{1}{\kappa_n \rho_n^2} \left(e^{|2\pi\rho_n x|} - |2\pi\rho_n x| - \frac{1}{2}|2\pi\rho_n x|^2 \right) \\ &\leqslant \frac{1}{\kappa_n \rho_n^2} \frac{(2\pi\rho_n x)^3}{6} (1 + o(1)) = o(\kappa_n^{-1}), \qquad n \to \infty. \end{aligned}$$

Hence,

$$\widehat{P}_n^{\lfloor\kappa_n\rfloor}(x) = \left(1 - \frac{2\pi^2 x^2}{\kappa_n} + \mathcal{E}_n(x)\right)^{\lfloor\kappa_n\rfloor} \to e^{-2\pi^2 x^2}, \qquad n \to \infty,$$

at least at each point x. Further, by a condition of Theorem 2.5,

$$(1 + C_1 \sigma_n^2 x^2)^{1/2} \widehat{p}_n(x) \leqslant 1,$$

from which it follows that

$$|\widehat{P}_n(x)| \leqslant \frac{1}{\sqrt{1 + x^2/(C_1\kappa_n)}}\,.$$

Therefore,

$$\|\widehat{P}_n^{\lfloor\kappa_n\rfloor} - e^{-2\pi^2x^2}\|_1 \leqslant \int_{-a}^{+a} |\widehat{P}_n^{\lfloor\kappa_n\rfloor}(x) - e^{-2\pi^2x^2}|\,dx + \int_{|x|>a} \left(1 + \frac{x^2}{C_1\kappa_n}\right)^{-\lfloor\kappa_n\rfloor/2} dx. \tag{6.1}$$

The first term tends to zero in view of the pointwise convergence, and the second is infinitesimal uniformly with respect to n, because for any k

$$\mathcal{I} := \int_a^\infty \left(1 + \frac{x^2}{C_1 k}\right)^{-k/2} dx = \sqrt{C_1}\,\frac{\sqrt{k}}{k-1}\left(\frac{\sqrt{k}}{a}\right)^{k-1} \cdot \mathbf{F}\left(\frac{k-1}{2}, \frac{k}{2}; \frac{k+1}{2}; -\frac{k}{a^2}\right),$$

where

$$\mathbf{F}(a, b; c; z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

is the hypergeometric function, and hence

$$\begin{aligned}\mathcal{I} &= \sqrt{C_1}\,\frac{\sqrt{k}}{k-1}\,\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{1}{\sqrt{t(1-t)}}\left(1 + \frac{a^2}{kt}\right)^{-(k-1)/2} dt \\ &\lesssim \sqrt{\frac{C_1}{2\pi}} \int_0^1 \frac{e^{-a^2/(2t)}}{\sqrt{t(1-t)}}\,dt = \sqrt{C_1} \int_a^\infty e^{-z^2/2}\,dz,\end{aligned}$$

which is what was to be proved.

Lemma 6.3. *If $a \geqslant \kappa_n$, then $\|p_n^{*a} - h_n^{-1}\|_\infty \leqslant h_n^{-1} 2^{-\lfloor a/\kappa_n\rfloor}$.*

Proof. We first make two general observations. Let us consider the densities ρ_1 and ρ_2 of some measures on the circle $\mathbb{T}$, assuming that $\|\rho_1 - 1\|_\infty \leqslant \varepsilon$. Then by the integral mean value theorem,

$$\|\rho_1 * \rho_2 - 1\|_\infty = \max_t \left|\int_{\mathbb{T}} (\rho_1(u) - 1)\rho_2(t-u)\,du\right| = \max_t |\rho_1(\widetilde{u}_t) - 1| \leqslant \varepsilon.$$

Suppose now that $\|\rho_1 - 1\|_\infty \leqslant \varepsilon_1$ and $\|\rho_2 - 1\|_\infty \leqslant \varepsilon_2$. Let $\delta_i(t) := \rho_i(t) - 1$. Then $\rho_1 * \rho_2 = (1+\delta_1) * (1+\delta_2) = 1 + \delta_1 * \delta_2$ and

$$\|\rho_1 * \rho_2 - 1\|_\infty = \|\delta_1 * \delta_2\|_\infty = \max_t \left| \delta_1(\widetilde{u}_t) \int_{\mathbb{T}} \delta_2(w)\,dw \right| \leqslant \varepsilon_1\varepsilon_2.$$

As above, we consider the density $P_n(t) = h_n p_n(\overline{\xi}_n + t h_n)$. Then the assertion of the lemma is equivalent to the inequality $\|P_n^{*a} - 1\|_\infty \leqslant 2^{\lfloor a/\kappa_n \rfloor}$. We have already shown that $\|P_n^{*\lfloor \kappa_n \rfloor} - 1\|_\infty \leqslant 2^{-1}$. Representing a in the form $a = \lfloor a/\lfloor \kappa_n \rfloor \rfloor \lfloor \kappa_n \rfloor + r$, we get that

$$\|P_n^{*a} - 1\|_\infty \leqslant \|P_n^{*\lfloor \kappa_n \rfloor} - 1\|_\infty^{\lfloor a/\lfloor \kappa_n \rfloor \rfloor} \leqslant 2^{-\lfloor a/\kappa_n \rfloor},$$

which is what was to be proved.

We consider the random variables $\Phi_n(0)$ and $\Phi_n(A)$. Let

$$\overline{m}_n := \frac{h_n}{h_n + \overline{\xi}_n}\,.$$

Note that the distribution of $\Phi_n(t)$ coincides with the measure $(1 - \overline{m}_n)\boldsymbol{\delta}_0 + \overline{m}_n \lambda_n$, where λ_n is normalized Lebesgue measure on I_n. We denote the density of the absolutely continuous component of their joint distribution by $p_n^\star(x, y; A)$.

Let $\widetilde{\Phi}_n(t)$ be the random variable taking values in $I_n \sqcup \mathbb{R}_+^2$ as follows: $\widetilde{\Phi}_n(t) := \Phi_n(t)$ if $\Phi_n(t) \neq 0$, and $\widetilde{\Phi}_n(t) := (l^-, l^+)$ if $\Phi_n(t) = 0$ and $[t - l^-, t + l^+]$ is the connected component of $\Phi_n^{-1}\{0\}$ containing t. In § 2 we verified that $\widetilde{\Phi}_n(t)$ is a Markov process. Let $\widetilde{p}_n^\star(\,\cdot\,, \cdot\,; A)$ denote the joint density of the random variables $\widetilde{\Phi}_n(0)$ and $\widetilde{\Phi}_n(A)$. Then it is clear that $p_n^\star(\,\cdot\,, \cdot\,; A) = \widetilde{p}_n^\star(\,\cdot\,, \cdot\,; A)|_{I_n \times I_n}$. Let $\widetilde{p}_n$ be the function on $I_n \sqcup \mathbb{R}_+^2$ given by

$$\widetilde{p}_n(x) := h_n^{-1}\overline{m}_n, \qquad x \in I_n;$$
$$\widetilde{p}_n(l^\pm) = h_n^{-1}\overline{m}_n p_n(l^- + l^+), \qquad l^\pm \in \mathbb{R}_+^2.$$

Theorem 6.4. *If $A \geqslant \kappa_n h_n$, then*

$$\|\widetilde{p}_n^\star(\widetilde{x}, \widetilde{y}; A) - \widetilde{p}_n(\widetilde{x})\,\widetilde{p}_n(\widetilde{y})\|_\infty \leqslant e^{c - \omega A/(\kappa_n h_n)} \widetilde{p}_n(\widetilde{x})\widetilde{p}_n(\widetilde{y})$$

and, in particular,

$$\|p_n^\star(x, y; A) - h_n^{-2}\overline{m}_n^2\|_\infty \leqslant h_n^{-2} e^{c - \omega A/(\kappa_n h_n)},$$

where $c = \ln 2$ and $\omega = \ln 2$.

Proof. We begin by studying the asymptotic behaviour of the density $\widetilde{p}_n^\star(\widetilde{x}, \widetilde{y}; A)$ as $n \to \infty$ for $A \sim \kappa_n h_n$. Let

$$p_{\mathrm{g}}^{(\alpha)}(t) := \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\biggl(-\frac{(t + k(1+\alpha))^2}{2}\biggr).$$

Lemma 6.5. *Let $A=\kappa_n h_n - v$, $0\leqslant v\leqslant 2h_n$. Then*

$$\left\| p_n^{\star}(x,y;A) - h_n^{-2}\overline{m}_n p_{\mathrm{g}}^{(h_n^{-1}\overline{\xi}_n)}\left(\frac{x-y}{h_n}\right)\right\|_\infty = o(h_n^{-2}), \qquad x,y\in I_n.$$

Proof of Lemma 6.5. Assume that $\Phi_n(0)=x\neq 0$. We number the open intervals making up the set $\Phi_n^{-1}(I_n\setminus\{0\})$ from left to right, assigning the number 0 to the interval containing the point 0. Let $\mathcal{I}(k)$ be the kth open interval. We also number the 'insertions' — the closed intervals making up the set $\Phi_n^{-1}\{0\}$ — from left to right, beginning with the first located to the right of the point $t=0$. In our case the point $t=0$ itself does not belong to $\Phi_n^{-1}\{0\}$. We note that the lengths s_j of the insertions are independent under the condition $\Phi_n(0)=x$.

We consider the event

$$\mathcal{S}_k = \left\{(h_n - x) + (k-1)h_n + \sum_{j=1}^{k} s_j \in [A-h_n, A]\right\} = \{\Phi_n(A)\neq 0 \wedge A\in\mathcal{I}(k)\}$$

corresponding to the interval $[0,A]$ containing exactly k insertions, and we note that $\mathcal{S}_k$ does not depend on the σ-algebra generated by $s_{k+1}, s_{k+2}, \dots$. Hence, the conditional joint density of the random variables $s_1,\dots,s_k$ with respect to the set $\mathcal{S}_k$ is the function $p_n(s_1)\cdots p_n(s_k)/\mathsf{P}(\mathcal{S}_k)$. Therefore, the density of $\Phi_n(A)=y$ under the condition $\mathcal{S}_k$ has the form

$$\frac{p_n^{*k}(A-kh_n+x-y)}{\mathsf{P}(\mathcal{S}_k)}.$$

Let

$$\mathcal{T}_{n,k} := [A-(k+1)h_n + x, A-kh_n+x]$$

be the interval in which the argument of the convolution power $p_n^{*k}(A-kh_n+x-y)$ varies. Thus,

$$p_n^{\star}(x,y;A) = h_n^{-1}\overline{m}_n\sum_{k=0}^{\infty} p_n^{*k}(A-kh_n+x-y).$$

Let $\lambda_{n,k} := k/\kappa_n$.

Lemma 6.6. *There exist $r_n\to\infty$ and $\varepsilon_n\to 0$ such that*

$$\left\| p_n^{*k}(t) - \frac{1}{\sqrt{2\pi}\,\lambda_{n,k}h_n}\exp\biggl(-\frac{(t-k\overline{\xi}_n)^2}{2\lambda_{n,k}^2h_n^2}\biggr)\right\|_\infty^{(\mathcal{T}_{n,k})} \leqslant \varepsilon_n \min_{t\in\mathcal{T}_{n,k}} \frac{1}{\sqrt{2\pi}\,h_n}\exp\biggl(-\frac{(t-k\overline{\xi}_n)^2}{2h_n^2}\biggr)$$

if $k\in K_n := \{k : |k-\kappa_n|\leqslant r_n\}$, where $\|\cdot\|_\infty^{(\mathcal{T}_{n,k})}$ denotes the uniform norm on the closed interval $\mathcal{T}_{n,k}$. The sequences r_n and ε_n can be chosen so that

$$\left\| \frac{1}{\sqrt{2\pi}\,\lambda_{n,k}h_n}\exp\biggl(-\frac{(t-k\overline{\xi}_n)^2}{2\lambda_{n,k}^2h_n^2}\biggr) - \frac{1}{\sqrt{2\pi}\,h_n}\exp\biggl(-\frac{(t-k\overline{\xi}_n)^2}{2h_n^2}\biggr)\right\|_\infty^{(\mathcal{T}_{n,k})} \leqslant \varepsilon_n \min_{t\in\mathcal{T}_{n,k}} \frac{1}{\sqrt{2\pi}\,h_n}\exp\biggl(-\frac{(t-k\overline{\xi}_n)^2}{2h_n^2}\biggr).$$

Proof of Lemma 6.6. Let $P_n(t) := p_n(th_n - \overline{\xi}_n)$ and $\widehat{P}_n(x) := \int_{\mathbb{R}} e^{-2\pi ixt} p_n(t)\,dt$. As earlier, we represent the function $\widehat{P}_n(x)$ in the form

$$\widehat{P}_n(x) = 1 - \frac{2\pi^2 x^2}{\kappa_n} + \mathcal{E}_n(x), \qquad \kappa_n = \frac{h_n^2}{\sigma_n^2}\,,$$

where $|\mathcal{E}_n(x)| = o(\kappa_n^{-1})$. Then for any $x \in \mathbb{R}$

$$\widehat{P}_n^k(x) \sim e^{-2\pi^2\lambda_{n,k}^2 x^2}, \qquad n \to \infty, \quad \lambda_{n,k} \to 1.$$

Repeating the argument in the proof of Lemma 6.2, we establish the pointwise convergence

$$p_n^{*k}(t) \to \frac{1}{\sqrt{2\pi}\,\lambda_{n,k} h_n} \exp\biggl(-\frac{(t - k\overline{\xi}_n)^2}{2\lambda_{n,k}^2 h_n^2}\biggr),$$

and hence the uniform convergence on the intervals $[(\kappa_n - r_n)h_n, (\kappa_n + r_n)h_n]$ for some slowly increasing sequence r_n, and this is what was to be proved.

Since $r_n \to \infty$, we have $\mathsf{P}\{k \in K_n\} \to 1$ as $n \to \infty$. We represent $p_n^\star(x, y; A)$ in the form $\mathcal{I}_0 + \mathcal{I}_1$, where

$$\mathcal{I}_0 = h_n^{-1}\overline{m}_n \sum_{k \in K_n} p_n^{*k}(A - kh_n + x - y),$$

and $\mathcal{I}_1$ is the sum over the complement of the set K_n. Let us estimate $\mathcal{I}_1$. It is not hard to show that

$$\|p_n^{*k}\|_\infty = O(\sigma_n^{-1}\kappa_n^{-1/2}) = O(h_n^{-1}), \qquad \|(p_n^{*k})'\|_\infty = O(\sigma^{-2}\kappa_n^{-1}) = O(h_n^{-2}),$$

from which it follows that $p_n^{*k}(A - kh_n + x - y)/\mathsf{P}(\mathcal{S}_k) = O(h_n^{-1})$. Hence,

$$\mathcal{I}_1 = O(h_n^{-2}(1 - \mathsf{P}\{k \in K\})) = o(h_n^{-2}).$$

Then

$$\begin{aligned}
&\biggl\|p_n^\star(x, y; A) - h_n^{-2}\overline{m}_n p_{\mathrm{g}}^{(h_n^{-1}\overline{\xi}_n)}\biggl(\frac{x-y}{h_n}\biggr)\biggr\|_\infty \\
&\qquad \leqslant \biggl\|\mathcal{I}_0 - h_n^{-2}\overline{m}_n p_{\mathrm{g}}^{(h_n^{-1}\overline{\xi}_n)}\biggl(\frac{x-y}{h_n}\biggr)\biggr\|_\infty + \mathcal{I}_1 \\
&\qquad \leqslant h_n^{-1} \sum_{k \in K_n} \biggl\|p_n^{*k}(t) - \frac{1}{\sqrt{2\pi}\,\lambda_{n,k} h_n} \exp\biggl(-\frac{(t - k\overline{\xi}_n)^2}{2\lambda_{n,k}^2 h_n^2}\biggr)\biggr\|_\infty \\
&\qquad\quad + h_n^{-1} \sum_{k \in K_n} \biggl\|\frac{1}{\sqrt{2\pi}\,\lambda_{n,k} h_n} \exp\biggl(-\frac{(t - k\overline{\xi}_n)^2}{2\lambda_{n,k}^2 h_n^2}\biggr) \\
&\qquad\quad - \frac{1}{\sqrt{2\pi}\,h_n} \exp\biggl(-\frac{(t - k\overline{\xi}_n)^2}{2h_n^2}\biggr)\biggr\|_\infty + \mathcal{I}_1 \leqslant \mathrm{const} \cdot \varepsilon_n h_n^{-2} + \mathcal{I}_1 = o(h_n^{-2}),
\end{aligned}$$

and Lemma 6.5 is proved.

Let $\widetilde{p}^\star_n(\widetilde{x},\widetilde{y};A) = \widetilde{p}_n(\widetilde{x})\widetilde{p}_n(\widetilde{y})(1+\delta^\star_n(\widetilde{x},\widetilde{y};A))$. We assume first that $A=\kappa_n h_n$ and we show that

$$\|\delta^\star_n(\,\cdot\,,\cdot\,;A)\|_\infty \leqslant \frac{1}{2}$$

for sufficiently large n. Lemma 6.5 asserts that

$$\left\|\delta^\star(x,y,;A) - \left(p_{\mathrm{g}}^{(h_n^{-1}\overline{\xi}_n)}\left(\frac{x-y}{h_n}\right)-1\right)\right\|_\infty = o(1).$$

It is easy to verify that $\|p_{\mathrm{g}}^{(0)}((x-y)/h_n)-1\|_\infty < 1/2$ and, moreover, that for sufficiently small α

$$\left\|\frac{\partial}{\partial\alpha}p_{\mathrm{g}}^{(\alpha)}(t)\right\|_\infty \leqslant \mathrm{const},$$

where the constant does not depend on α. Hence, since $h_n^{-1}\overline{\xi}_n \to 0$, we have $\|\delta^\star_n(\,\cdot\,,\cdot\,;A)|_{I_n\times I_n}\|_\infty \leqslant 1/2$ starting with some n. Furthermore,

$$\widetilde{p}^\star_n(x,l^\pm;A) = \widetilde{p}^\star_n(x,h_n-l^-;A-h_n)p_n(l^-+l^+) = \frac{\widetilde{p}^\star_n(x,h_n-l^-;A-h_n)}{h_n^{-1}\overline{m}_n}\widetilde{p}_n(l^\pm),$$

from which it follows that

$$\|\delta^\star_n(x,l^\pm;A)\|_\infty \leqslant \|\delta^\star_n(x,h_n-l^-;A-h_n)\|_\infty \leqslant \frac{1}{2}\,.$$

The estimates of $\delta^\star_n(l^\pm,x;A)$ and $\delta^\star_n(l_1^\pm,l_2^\pm;A)$ for $A=\kappa_n h_n$ are obtained similarly.

Using the Chapman–Kolmogorov formula, we compute $\widetilde{p}^\star_n(\widetilde{x},\widetilde{y};A_1+A_2)$:

$$\widetilde{p}^\star_n(\widetilde{x},\widetilde{y};A_1+A_2) = \widetilde{p}_n(\widetilde{x})\int_{I_n\sqcup\mathbb{R}^2_+}\frac{\widetilde{p}^\star_n(\widetilde{x},\widetilde{z};A_1)\widetilde{p}^\star_n(\widetilde{z},ty;A_2)}{\widetilde{p}_n(\widetilde{x})\widetilde{p}_n(\widetilde{z})}\,d\widetilde{z}.$$

Note that

$$\int_{I_n\sqcup\mathbb{R}^2_+}\delta^\star_n(\widetilde{x},\widetilde{z};A_1)\widetilde{p}_n(\widetilde{z})\,d\widetilde{z} = \int_{I_n\sqcup\mathbb{R}^2_+}\frac{p^\star_n(\widetilde{x},\widetilde{z};A_1)}{\widetilde{p}_n(\widetilde{x})}\,d\widetilde{z}-1=0,$$

since by definition

$$\int_{I_n\sqcup\mathbb{R}^2_+}\widetilde{p}^\star_n(\widetilde{x},\widetilde{z};A_1)\,d\widetilde{z} = \widetilde{p}_n(\widetilde{x}).$$

We have

$$\begin{aligned}&\frac{\widetilde{p}_n(\widetilde{x})\widetilde{p}_n(\widetilde{y})(1+\delta^\star_n(\widetilde{x},\widetilde{y};A_1+A_2))}{\widetilde{p}_n(\widetilde{x})}\\ &\qquad=\int_{I_n\sqcup\mathbb{R}^2_+}\frac{\widetilde{p}_n(\widetilde{x})\widetilde{p}_n(\widetilde{z})^2\widetilde{p}_n(\widetilde{y})(1+\delta^\star_n(\widetilde{x},\widetilde{z};A_1))(1+\delta^\star_n(\widetilde{z},\widetilde{y};A_2))}{\widetilde{p}_n(\widetilde{x})\,\widetilde{p}_n(\widetilde{z})}\,d\widetilde{z},\end{aligned}$$

from which we see that

$$1+\delta_n^\star(\widetilde{x},\widetilde{y};A_1+A_2)=1+\int_{I_n\sqcup\mathbb{R}_+^2}\delta_n^\star(\widetilde{x},\widetilde{z};A_1)\widetilde{p}_n(\widetilde{z})\,d\widetilde{z}$$
$$+\int_{I_n\sqcup\mathbb{R}_+^2}\delta_n^\star(\widetilde{z},\widetilde{y};A_2)\widetilde{p}_n(\widetilde{z})\,d\widetilde{z}+\int_{I_n\sqcup\mathbb{R}_+^2}\delta_n^\star(\widetilde{x},\widetilde{z};A_1)\delta_n^\star(\widetilde{z},\widetilde{y};A_2)\,d\widetilde{z},$$

and finally,

$$\delta_n^\star(\widetilde{x},\widetilde{y};A_1+A_2)=\int_{I_n\sqcup\mathbb{R}_+^2}\delta_n^\star(\widetilde{x},\widetilde{z};A_1)\delta_n^\star(\widetilde{z},\widetilde{y};A_2)\,d\widetilde{z}.$$

Thus,

$$\|\delta_n^\star(\widetilde{x},\widetilde{y};A)\|_\infty\leqslant 2^{1-\lfloor A/(\kappa_n h_n)\rfloor},$$
$$\|\widetilde{p}_n^\star(\widetilde{x},\widetilde{y};A)-\widetilde{p}_n(\widetilde{x})\,\widetilde{p}_n(\widetilde{y})\|_\infty\leqslant e^{c-\omega A/(\kappa_n h_n)}\widetilde{p}_n(\widetilde{x})\,\widetilde{p}_n(\widetilde{y}),$$

where $c=2\ln 2$, $\omega=\ln 2$, and $A\geqslant\kappa_n h_n$. Theorem 6.4 is proved.

Let $\boldsymbol{i}\in\mathcal{B}(\widetilde{I}_n)$ and $A\in[0,h_{n+1}]$. We consider the random process

$$\eta_{\boldsymbol{i}}(t):=\begin{cases}1 & \text{if } \theta_n(t,A)\in\boldsymbol{i} \quad\text{and}\quad \Phi_n(t),\Phi_n(A+t)\neq 0,\\ 0 & \text{otherwise,}\end{cases}$$
$$\theta_n(t,A):=\Phi_n(A+t)-\Phi_n(t),\qquad 0\leqslant t\leqslant h_{n+1}-A.$$

The expectation of the random variable $\eta_{\boldsymbol{i}}(t)$ can be expressed by the formula

$$\overline{\eta}_{\boldsymbol{i}}=\int_{D_n(\widetilde{\Delta}_n)}p_n^\star(x,y;A)\,dx\,dy,$$

where $D_n(B)=\{(x,y)\in I_n\times I_n : y-x\in B\}$ for $B\subseteq\widetilde{I}_n$.

We show that the random process $\eta_{\boldsymbol{i}}$ is ψ-mixing, where ψ is an exponentially decreasing function. Let

$$t''\geqslant t'+A+l'.$$

We introduce the following notation. Let $\widetilde{x}_0$, $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{x}_3$ and $\widetilde{y}_0$, $\widetilde{y}_1$, $\widetilde{y}_2$, $\widetilde{y}_3$ be the values of the function $\widetilde{\Phi}_n$ at the points $t'+A$, $t'+A+l'$, $t'+l'$, t' and $t''+l''$, t'', $t''+A$, $t''+A''+l''$, respectively. Then

$$\mathsf{P}(w'\wedge w'')=\int W'(\widetilde{x})W''(\widetilde{y})P(\widetilde{x}\mid\widetilde{x}_1)P(\widetilde{y}\mid\widetilde{y}_1)P_1(\widetilde{x}_1,\widetilde{y}_1)\,d\widetilde{x}\,d\widetilde{y},$$
$$\mathsf{P}(w')=\int W'(\widetilde{x})P(\widetilde{x}\mid\widetilde{x}_1)\widetilde{p}_n(\widetilde{x}_1)\,d\widetilde{x},\qquad \mathsf{P}(w'')=\int W''(\widetilde{y})P(\widetilde{y}\mid\widetilde{y}_1)\widetilde{p}_n(\widetilde{y}_1)\,d\widetilde{y},$$

where $P(\widetilde{x}_1,\widetilde{y}_1)$ is the density of the joint distribution of $\widetilde{x}_1$ and $\widetilde{y}_1$:

$$P(\widetilde{x}_1,\widetilde{y}_1)=\widetilde{p}_n^\star(\widetilde{x}_1,\widetilde{y}_1;t''-(t'+l'+A)).$$

We recall that

$$\|P(\widetilde{x}_1, \widetilde{y}_1) - \widetilde{p}_n(\widetilde{x}_1)\widetilde{p}_n(\widetilde{y}_1)\|_\infty \leqslant \widetilde{p}_n(\widetilde{x}_1)\widetilde{p}_n(\widetilde{y}_1) e^{c-\omega(t''-(t'+l'+A))/(\kappa_n h_n)}$$

if $t'' - (t' + l' + A) \geqslant \kappa_n h_n$. Hence,

$$|\mathsf{P}(w' \wedge w'') - \mathsf{P}(w')\mathsf{P}(w'')| \leqslant \mathsf{P}(w')\mathsf{P}(w'') e^{c-\omega(t''-(t'+l'+A))/(\kappa_n h_n)}, \tag{6.2}$$

and therefore the process $\eta_{\boldsymbol{i}}$ is ψ-mixing with the function

$$\psi(t) = \begin{cases} e^{c-\omega(t-A)/(\kappa_n h_n)} & \text{if } t \geqslant A + \kappa_n h_n, \\ +\infty & \text{otherwise.} \end{cases}$$

We assume now that

$$t' < t' + l' < t'' < t'' + l'' < t' + A < t' + A + l' < t'' + A < t'' + A + l'',$$
$$0 \leqslant t' < t'' + l'' \leqslant \frac{A}{2}.$$

The density of the joint distribution of x_0 and y_0 will be denoted by

$$P_0(\widetilde{x}_0, \widetilde{y}_0) = \widetilde{p}_n^\star(\widetilde{y}_0, \widetilde{x}_0; t' + A - (t'' + l'')).$$

We consider the conditional densities

$$P_0'(\widetilde{x}_1 \mid \widetilde{x}_0),\ P_0''(\widetilde{y}_1 \mid \widetilde{y}_0),\ P_1'(\widetilde{x}_2 \mid \widetilde{y}_1),\ P_1''(\widetilde{y}_2 \mid \widetilde{x}_1),\ P_2'(\widetilde{x}_3 \mid \widetilde{x}_2),\ P_2''(\widetilde{y}_3 \mid \widetilde{y}_2).$$

Let $W'(\widetilde{x}) = \mathsf{P}(w' \mid \widetilde{x})$ and $W''(\widetilde{y}) = \mathsf{P}(w'' \mid y)$. Since $\widetilde{\Phi}_n$ is a Markov process,

$$\mathsf{P}(w' \wedge w'') = \int P_0 P_0' P_0'' P_1' P_1'' P_2' P_2'' W'(\widetilde{x}) W''(\widetilde{y})\, d\widetilde{x}\, d\widetilde{y},$$
$$\mathsf{P}(w') = \int P_0 P_0' P_0'' P_1' P_2' W'(\widetilde{x})\, d\widetilde{x}\, d\widetilde{y},$$
$$\mathsf{P}(w'') = \int P_0 P_0' P_0'' P_1'' P_2'' W''(\widetilde{y})\, d\widetilde{x}\, d\widetilde{y}.$$

To make the expressions less cumbersome, we omit the arguments of $P_0, P_0', P_0'', \dots$. Let

$$\mathcal{I}_1 = \int \widetilde{p}_n(\widetilde{x}_0)\widetilde{p}_n(\widetilde{y}_0) P_0' P_0'' P_1' P_1'' P_2' P_2'' W'(\widetilde{x}) W''(\widetilde{y})\, d\widetilde{x}\, d\widetilde{y},$$
$$\mathcal{I}_1' = \int \widetilde{p}_n(\widetilde{x}_0)\widetilde{p}_n(\widetilde{y}_0) P_0' P_0'' P_1' P_2' W'(\widetilde{x})\, d\widetilde{x}\, d\widetilde{y},$$
$$\mathcal{I}_1'' = \int \widetilde{p}_n(\widetilde{x}_0)\widetilde{p}_n(\widetilde{y}_0) P_0' P_0'' P_1'' P_2'' W''(\widetilde{y})\, d\widetilde{x}\, d\widetilde{y}$$

and

$$\mathcal{I}_2 = \int \widetilde{p}_n(\widetilde{x}_0)\widetilde{p}_n(\widetilde{y}_0)\widetilde{p}_n(\widetilde{x}_1)\widetilde{p}_n(\widetilde{y}_1)P_0'P_0''P_2'P_2''W'(\widetilde{x})W''(\widetilde{y})\,d\widetilde{x}\,d\widetilde{y},$$

$$\mathcal{I}_2' = \int \widetilde{p}_n(\widetilde{x}_0)\widetilde{p}_n(\widetilde{y}_0)\widetilde{p}_n(\widetilde{x}_1)\widetilde{p}_n(\widetilde{y}_1)P_0'P_0''P_2'W'(\widetilde{x})\,d\widetilde{x}\,d\widetilde{y},$$

$$\mathcal{I}_2'' = \int \widetilde{p}_n(\widetilde{x}_0)\widetilde{p}_n(\widetilde{y}_0)\widetilde{p}_n(\widetilde{x}_1)\widetilde{p}_n(\widetilde{y}_1)P_0'P_0''P_2'W''(\widetilde{y})\,d\widetilde{x}\,d\widetilde{y}.$$

We first use the fact that the joint distribution of $\widetilde{x}$ and $\widetilde{y}$ is almost uniform if the interval between the corresponding moments of time is sufficiently large:

$$|\mathsf{P}(w'\wedge w'') - \mathcal{I}_1| \leqslant \mathcal{I}_1\mathcal{E}_0, \qquad |\mathsf{P}(w') - \mathcal{I}_1'| \leqslant \mathcal{I}_1'\mathcal{E}_0, \qquad |\mathsf{P}(w'') - \mathcal{I}_1''| \leqslant \mathcal{I}_1''\mathcal{E}_0,$$

where $\mathcal{E}_0 := e^{c-\omega(t'+A-(t''+l''))/(\kappa_n h_n)}$ and $t'+A-(t''+l'') \geqslant \kappa_n h_n$. Noting that $\mathcal{I}_2 = \mathcal{I}_2'\mathcal{I}_2''$, we get that

$$|\mathsf{P}(w'\wedge w'') - \mathsf{P}(w')\mathsf{P}(w'')| \leqslant (\mathcal{I}_1 + \mathsf{P}(w')\mathcal{I}_1'' + \mathcal{I}_2''\mathcal{I}_1')\mathcal{E}_0 + (\mathcal{I}_2 + \mathsf{P}(w')\mathcal{I}_2'' + \mathcal{I}_2''\mathcal{I}_2')\mathcal{E}_1,$$

where $\mathcal{E}_1 := e^{c-\omega(t''-(t'+l'))/(\kappa_n h_n)}$ and $t''-(t'+l') \geqslant \kappa_n h_n$. From this it is not hard to obtain the relation

$$|\mathsf{P}(w'\wedge w'') - \mathsf{P}(w')\mathsf{P}(w'')| \leqslant (C_0\mathcal{E}_0 + C_1\mathcal{E}_1)\mathsf{P}(w')\mathsf{P}(w'').$$

In particular, if $\kappa_n h_n \leqslant |(t''+l'')-t'| \leqslant A/2$, then

$$\begin{aligned}|\mathsf{P}(w'\wedge w'') - \mathsf{P}(w')\mathsf{P}(w'')| &\leqslant (C_0 + C_1)\mathcal{E}_0\mathsf{P}(w')\mathsf{P}(w'')\\ &= \mathsf{P}(w')\mathsf{P}(w'')e^{c_2-\omega(t''-(t'+l'))/(\kappa_n h_n)}.\end{aligned} \tag{6.3}$$

We need this estimate in § 7.

§ 7. Proof of the mixing property

In this section we prove Theorem 2.5. Namely, we establish that the flow $\mathbf{T}_\omega$ constructed is P-almost surely mixing. To this end it suffices to show that for any n_0 and any functions $f,g \in C^1(I_{n_0})$ with the properties $\Theta f = \Theta g = f(0) = g(0) = 0$ we have

$$R(A) = R_{f,g}(A) := \langle \widehat{T}^A f_\star, g_\star\rangle \to 0, \qquad A\to\infty$$

with probability 1, where $f_\star$ and $g_\star$ are the functions on X corresponding to f and g:

$$f_\star(x) = f(\varphi_n(x)), \qquad g_\star(x) = g(\varphi_n(x)).$$

Indeed, we can choose a countable set $\mathcal{F}$ of functions of such form for which finite linear combinations of them are dense in the space $L^2(X,\mu)$. If $R_{f,g}(A)\to 0$ for all pairs of functions $f,g\in\mathcal{F}$, then the flow is mixing; and mixing takes place on all pairs f, g with probability 1, because the set $\mathcal{F}$ is countable.

Accordingly, let us fix two functions $f,g\in C^1(I_{n_0})$ such that

$$\|f\|_\infty, \|g\|_\infty \leqslant 1, \qquad \|f'\|_\infty, \|g'\|_\infty \leqslant b < \infty, \qquad \Theta f = \Theta g = f(0) = g(0) = 0.$$

Since the functions $f_\star$ and $g_\star$ are $\mathcal{A}(\varphi_n)$-measurable, they can be regarded as functions on the space $(X, \mathcal{A}(\varphi_n), \mu)$, which is isomorphic in a natural way to the space $(I_n, \mathcal{B}(I_n), \mu_n)$. More precisely, let

$$f_{(n)}(t) = f(\varphi_{n,n_0}(t)),$$
$$g_{(n)}(t) = g(\varphi_{n,n_0}(t)), \qquad t \in I_n.$$

Let $I_n^\circ := (0, h_n)$ and $J_n^\circ = \overleftarrow{\varphi}_n^{-1} I_n^\circ$. The set J_n° is the union of the intervals of the form $l_{n,k} + I_n^\circ$, and the set $S^A J_n^\circ \cap J_n^\circ$ is a union of intervals which we denote by $\mathcal{I}_{n,j}^\circ = u_{n,j} + \mathcal{I}_n^\circ(A_j)$, where

$$\mathcal{I}_n^\circ(A) := S^A(0, h_n) \cap (0, h_n) = \begin{cases} (0, h_n - A), & 0 \leqslant A \leqslant h_n, \\ (A, h_n), & -h_n \leqslant A \leqslant 0, \end{cases} \qquad A \in \widetilde{I}_{n+1}.$$

Let

$$\begin{aligned} \widetilde{R}_n(A) &:= \int_{\mathcal{I}_n^\circ(A)} f_{(n)}(t + A)\overline{g_{(n)}(t)}\, dt \\ &= \begin{cases} \dfrac{1}{h_n - |A|} \displaystyle\int_0^{h_n - A} f_{(n)}(t + A)\overline{g_{(n)}(t)}\, dt, & 0 \leqslant t \leqslant h_n, \\ \dfrac{1}{h_n - |A|} \displaystyle\int_A^{h_n} f_{(n)}(t + A)\overline{g_{(n)}(t)}\, dt, & -h_n \leqslant t \leqslant 0. \end{cases} \end{aligned}$$

The function $\widetilde{R}_n$ is defined on the set $\widetilde{I}_n := [-h_n, h_n]$, which it is sometimes convenient to identify with the circle $\mathbb{R}/2h_n\mathbb{Z}$. We remark that if $A \ll h_n$, then $R(A) \approx \widetilde{R}_n(A)$. We define a family of measures $\widetilde{\nu}_n(\,\cdot\,, A)$ on $\widetilde{I}_n$ as follows: for a Borel subset B of the closed interval $\widetilde{I}_n$ let

$$\widetilde{\nu}_n(B, A) := \frac{\ell}{h_{n+1} - |A|}\,,$$

where ℓ is the sum of the lengths of the intervals $\mathcal{I}_{n,j}^\circ$ such that $A_j \in B$. The discrete measure $\widetilde{\nu}_n$ is not normalized, but $\widetilde{\nu}_n(\widetilde{I}_n, A) \geqslant 2m_n - 1 \to 1$. From our assumptions about f and g it follows that

$$\widetilde{R}_{n+1}(A) = \int_{\widetilde{I}_n} \widetilde{R}_n(t)\, \widetilde{\nu}_n(dt, A). \tag{7.1}$$

Indeed, since $f(0) = g(0) = 0$, we have

$$\begin{aligned} \widetilde{R}_{n+1}(A) &= \frac{1}{h_{n+1} - |A_j|} \int_0^{h_{n+1} - A} f_{(n)}(t + A)\overline{g_{(n)}(t)}\, dt \\ &= \frac{1}{h_{n+1} - |A_j|} \sum_j \int_{\mathcal{I}_{n,j}^\circ} f_{(n)}(t + A)\overline{g_{(n)}(t)}\, dt \\ &= \frac{1}{h_{n+1} - |A_j|} \sum_j \int_{\mathcal{I}_n^\circ(A_j)} f(t + A_j)\overline{g(t)}\, dt \\ &= \sum_j \frac{\lambda(\mathcal{I}_n^\circ(A_j))}{h_{n+1} - |A_j|} \widetilde{R}_n(A_j) = \int_{\widetilde{I}_n} \widetilde{R}_n(t)\, \widetilde{\nu}_n(dt, A) \end{aligned}$$

for $A>0$. Let us consider the measures

$$\widetilde{\nu}_n^\bullet(B,A) := S^{-A}\widetilde{\nu}_n(B,A) = \widetilde{\nu}(S^{-A}B,A).$$

We note that when A varies within the limits of some bounded interval, the measures $\widetilde{\nu}_n^\bullet(\,\cdot\,,A)$ are concentrated on a finite set of points. We give a more precise description of the behaviour of the measure $\widetilde{\nu}_n^\bullet$. It has the representation

$$\widetilde{\nu}_n^\bullet(\,\cdot\,,A) = \sum_{z\in\mathcal{Z}} \mathfrak{a}(\,\cdot\,,A;\overline{\mathfrak{a}}_z,A_z),$$

where $\mathfrak{a}(\,\cdot\,,A;\overline{\mathfrak{a}}_z,A_z)$ is an '*atom*', that is, a measure of the form $p(A;A_z)\boldsymbol{\delta}_{\overline{\mathfrak{a}}_z}$, $A_z\in\widetilde{I}_{n+1}$, $\overline{\mathfrak{a}}_z\in\widetilde{I}_n$, with

$$p(A;A_z) = \frac{h_n}{h_{n+1}-|A|}\begin{cases} 1-h_n^{-1}|A-A_z| & \text{if } |A-A_z|\leqslant h_n, \\ 0 & \text{otherwise,}\end{cases}$$

and $\mathcal{Z}$ is some finite set. Each atom corresponds to a single interval $\mathcal{I}^\circ_{n,j}$. It is not hard to verify the following property of the function $p(A;A_z)$:

$$p(A;A_z)\leqslant p(ah_n;A_z)+p((a+1)h_n;A_z) \quad \text{if} \quad A\in[ah_n,(a+1)h_n].$$

This gives us that

$$\mathfrak{a}(\,\cdot\,,A;\overline{\mathfrak{a}},A_z)\leqslant \mathfrak{a}(\,\cdot\,,ah_n;\overline{\mathfrak{a}},A_z)+\mathfrak{a}(\,\cdot\,,(a+1)h_n;\overline{\mathfrak{a}},A_z),$$

and hence

$$\widetilde{\nu}_n^\bullet(\,\cdot\,,A)\leqslant \widetilde{\nu}_n^\bullet(\,\cdot\,,ah_n)+\widetilde{\nu}_n^\bullet(\,\cdot\,,(a+1)h_n).$$

Let

$$\mathcal{A}_n = (2\delta_n h_{n+1}, 2\delta_{n+1}h_{n+2}], \quad \text{where} \quad \delta_n = q_n^{-\chi}\ll 1.$$

We fix $A_0\in\mathcal{A}_n$ and note that $|\widetilde{R}_{n+2}(A_0)-R(A_0)|\leqslant 8\delta_{n+1}\to 0$. As before, we assign to A_0 the intersection $S^{A_0}J^\circ_{n+1}\cap J^\circ_{n+1}$, which is the union of the intervals $\mathcal{I}^\circ_{n+1,j}=u_{n+1,j}+\mathcal{I}^\circ_{n+1}(A_j)$. Let us show that the fraction of those A_j that belong to the set

$$\widetilde{\Delta}_{n+1} := [-h_{n+1},-(1-\delta_n)h_{n+1}]\cup[-\delta_n h_{n+1},\delta_n h_{n+1}]\cup[(1-\delta_n)h_{n+1},h_{n+1}],$$

is almost surely small (tends to zero). More precisely, we estimate the probability P_0 of the following event:

$$\widetilde{\nu}_{n+1}(\widetilde{\Delta}_{n+1},A_0)\leqslant 8v_{n+1}+16q_{n+1}^{-1} \qquad \forall\, A_0\in\mathcal{A}_n, \tag{7.2}$$

where $v_{n+1}\geqslant \delta_n h_{n+1}\|p_{n+1}\|_\infty$ is a sequence that tends to zero sufficiently slowly. We consider the sets $\boldsymbol{i}=\boldsymbol{i}(k)$ of the form

$$k\cdot 2\overline{\delta}_n h_{n+1}+([-h_{n+1},-(1-\overline{\delta}_n)h_{n+1}]\cup[-\overline{\delta}_n h_{n+1},\overline{\delta}_n h_{n+1}]\cup[(1-\overline{\delta}_n)h_{n+1},h_{n+1}]),$$

which form a covering of the circle $\widetilde{I}_{n+1}$, where $\overline{\delta}_n$ is the minimal positive number $\geqslant \delta_n$ such that $\overline{\delta}_n^{-1} \in \mathbb{N}$. The condition (7.2) holds if:

$$ah_{n+1} \in \mathcal{A}_n \Longrightarrow \widetilde{\nu}^{\bullet}_{n+1}(\boldsymbol{i}, ah_{n+1}) \leqslant 2v_{n+1} + 4q_{n+1}^{-1} \qquad \forall\, a \in \mathbb{N},\ \forall\, \boldsymbol{i}. \tag{7.3}$$

Indeed, suppose to begin with that $A_0 \geqslant h_{n+1}$, namely, suppose that $ah_{n+1} \leqslant A_0 < (a+1)h_{n+1}$. Then since any shift of $\widetilde{\Delta}_{n+1}$ is covered by two sets $\boldsymbol{i}'$ and $\boldsymbol{i}''$, we have

$$\begin{aligned}
\widetilde{\nu}^{\bullet}_{n+1}(S^{-A_0}\widetilde{\Delta}_{n+1}, A_0) &\leqslant \widetilde{\nu}^{\bullet}_{n+1}(\boldsymbol{i}', A_0) + \widetilde{\nu}^{\bullet}_{n+1}(\boldsymbol{i}'', A_0) \\
&\leqslant \widetilde{\nu}^{\bullet}_{n+1}(\boldsymbol{i}', ah_{n+1}) + \widetilde{\nu}^{\bullet}_{n+1}(\boldsymbol{i}', (a+1)h_{n+1}) \\
&\quad + \widetilde{\nu}^{\bullet}_{n+1}(\boldsymbol{i}'', ah_{n+1}) + \widetilde{\nu}^{\bullet}_{n+1}(\boldsymbol{i}'', (a+1)h_{n+1}) \\
&\leqslant 8v_{n+1} + 16\overline{q}_{n+1}^{-1},
\end{aligned}$$

where $n \in \mathbb{N}$. Suppose now that $2\delta_n h_{n+1} < A_0 < h_{n+1}$. In this case it suffices to use the same arguments, noting only that for the given A_0 the intervals $\boldsymbol{i}'$ and $\boldsymbol{i}''$ are not equal to $\widetilde{\Delta}_{n+1}$, and the measure $\widetilde{\nu}^{\bullet}_{n+1}(\,\cdot\,,0)$ is concentrated at the point 0.

Accordingly, we fix a and an interval $\boldsymbol{i}$, let $A = ah_{n+1}$, and consider the random process

$$\eta_{\boldsymbol{i}}(t) := \begin{cases} 1 & \text{if } \theta_{n+1}(t, A) \in \boldsymbol{i} \quad \text{and} \quad \Phi_{n+1}(t), \Phi_{n+1}(t+A) \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$
$$\theta_{n+1}(t, A) := \Phi_{n+1}(A+t) - \Phi_{n+1}(t), \qquad 0 \leqslant t \leqslant h_{n+2} - A.$$

It is obvious that

$$\widetilde{\nu}^{\bullet}_{n+1}(\boldsymbol{i}, A) = \widetilde{\nu}_{n+1}(\boldsymbol{i}, A) = \frac{1}{h_{n+2} - A} \int_0^{h_{n+2}-A} \eta_{\boldsymbol{i}}(t)\, dt + \mathcal{E}_{n+1},$$

where

$$|\mathcal{E}_{n+1}| \leqslant 4\overline{q}_{n-1}^{-1}.$$

The expectation of the random variable $\eta_{\boldsymbol{i}}(t)$ can be expressed by the formula

$$\overline{\eta}_{\boldsymbol{i}} = \int_{D_{n+1}(\widetilde{\Delta}_{n+1})} p^{\star}_{n+1}(x, y; A)\, dx\, dy,$$
$$D_{n+1}(B) = \{(x, y) \in I_{n+1} \times I_{n+1} : y - x \in B\}, \qquad B \subseteq \widetilde{I}_{n+1}.$$

In § 6 we proved that

$$\|p^{\star}_{n+1}(\,\cdot\,; A) - \overline{m}_{n+1}^2 h_{n+1}^{-2}\|_\infty \leqslant \|p^{\star}_{n+1}(\,\cdot\,; h_{n+1}) - \overline{m}_{n+1}^2 h_{n+1}^{-2}\|_\infty;$$

hence

$$\begin{aligned}
\overline{\eta}_{\boldsymbol{i}} &\leqslant \overline{m}_{n+1}^2 \widetilde{\lambda}_{n+1}(\boldsymbol{i}) + \|p^{\star}_{n+1}(\,\cdot\,; h_{n+1}) - \overline{m}_{n+1}^2 h_{n+1}^{-2}\|_\infty \cdot 4\delta_n h_n^2 \\
&\leqslant 2\overline{m}_{n+1}^2 \widetilde{\lambda}_{n+1}(\boldsymbol{i}) + \|p^{\star}_{n+1}(\,\cdot\,; h_{n+1})\|_\infty \cdot 4\delta_n h_n^2 \\
&\leqslant 2\overline{m}_n^2 \cdot 4\delta_n + \|\overline{m}_{n+1} h_{n+1}^{-1} p_{n+1}\|_\infty \cdot 4\delta_n h_n^2 \\
&\leqslant 8\delta_n + 4\|\delta_n h_{n+1} p_{n+1}\|_\infty \to 0.
\end{aligned}$$

The sequence v_n can be chosen so that $v_{n+1} \geqslant \overline{\eta}_{\boldsymbol{i}}$.

It has been proved that the process $\eta_{\boldsymbol{i}}$ is ψ-mixing with the function

$$\psi(t) = \begin{cases} e^{c_2 - \omega(t-A)/(\kappa_{n+1}h_{n+1})}, & t \geqslant t_0 := A + \kappa_{n+1}h_{n+1}, \\ \infty & \text{otherwise.} \end{cases}$$

We consider the set $\mathcal{I} := \{t : \eta_{\boldsymbol{i}}(t) = 1\}$. Let us apply Theorem 5.11 to the process $\eta_{\boldsymbol{i}}$, setting $\boldsymbol{\omega} = \omega/(\kappa_{n+1}h_{n+1})$, $T = h_{n+2} - A$, $\mathbf{c} = c_2 + \omega A/(\kappa_{n+1}h_{n+1})$, $\mathbf{h} = h_{n+1}$, $\boldsymbol{\rho} = 4$, and $\varepsilon = v_{n+1}$. We verify the conditions of the theorem. Obviously,

$$\#(\partial\mathcal{I} \cap [0,t]) \leqslant \frac{4t}{h_{n+1}}.$$

Further,

$$\boldsymbol{\omega}\tau = \frac{\omega}{\kappa_{n+1}h_{n+1}} \frac{v_{n+1}h_{n+1}}{16} = \frac{\omega v_{n+1}}{16\kappa_{n+1}} \leqslant \mathbf{c};$$
$$\frac{t_0}{\tau} = \frac{A}{\tau} + \frac{16\kappa_{n+1}}{v_{n+1}} \leqslant \frac{\ln(T/\tau) + c_2}{v_{n+1}\omega/(16\kappa_{n+1})} + \frac{A}{\tau},$$

where $\tau = \varepsilon\mathbf{h}/(4\boldsymbol{\rho}) = v_{n+1}h_{n+1}/16$, since it can be assumed without loss of generality that $T > e^{\omega}\tau$. The remaining conditions are obvious. We have

$$\begin{aligned} \mathsf{P}\{\widetilde{\nu}_{n+1}(\boldsymbol{i}, A) > 2v_{n+1} + 4q_{n+1}^{-1}\} &\leqslant \mathsf{P}\left\{\frac{1}{h_{n+2} - A}\int_0^{h_{n+2}-A} \eta_{\boldsymbol{i}}(t)\,dt - \overline{\eta}_{\boldsymbol{i}} > v_{n+1}\right\} \\ &\leqslant \exp\left(-c_3 v_{n+1}^2 \frac{\omega(h_{n+2} - A)}{\kappa_{n+1}h_{n+1}\big(\ln h_{n+2} - \frac{\ln(v_{n+1}h_{n+1})}{16} + 2c_2 + 2\frac{\omega A}{k_{n+1}h_{n+1}}\big)}\right) \\ &\leqslant \exp\left(-c_3 v_{n+1}^2 \frac{\omega(q_{n+1} - a)}{\kappa_{n+1}(\ln q_{n+1} - \ln v_{n+1}/16) + 2c_2 + 2\omega a}\right). \end{aligned}$$

By a condition of the theorem, $\kappa_{n+1} = o(q_{n+1}^{1-\chi-\epsilon})$. Moreover, $a \leqslant \delta_{n+1}q_{n+1} = q_{n+1}^{1-\chi}$. Hence,

$$\mathsf{P}\{\widetilde{\nu}_{n+1}(\boldsymbol{i}, A) > 2v_{n+1} + 4q_{n+1}^{-1}\} \leqslant \exp\left(-c_3 v_{n+1}^2 \frac{\omega q_{n+1}^{\chi}}{\varkappa(q_{n+1})}\right),$$

where $\varkappa(q_{n+1}) = o(q_{n+1}^{\varepsilon})$ for any $\varepsilon > 0$. Since this quantity decreases exponentially (with respect to q_n) and since the total number of pairs $(a, \boldsymbol{i})$ is bounded by a polynomial in q_n, the condition (7.2) holds almost surely starting with some n.

We assume that $A_0 \geqslant A_{00}$. It was shown above that for any such A_0 the correlation function $\widetilde{R}_{n+2}$ is determined mainly by the values of $\widetilde{R}_{n+1}$ at the points A_j with the property $\delta_n h_{n+1} \leqslant |A| \leqslant (1-\delta_n)h_{n+1}$, which are almost surely a majority of points. Our immediate goal is to prove that with probability 1

$$\sup_{\widetilde{I}_{n+1}\setminus\widetilde{\Delta}_{n+1}} |\widetilde{R}_{n+1}(A)| \to 1, \qquad n \to \infty.$$

To see this, we proceed as follows. We choose a sequence s_n tending to zero and show that $|\widetilde{R}_{n+1}(A)| \to 0$ almost surely for all points of the form $A = js_n$, $j \in \mathbb{Z}$. Then by the condition $\|f'\|_\infty \leqslant b$ we have

$$\left|\widetilde{R}_{n+1}(A) - \widetilde{R}_{n+1}\left(\left\lfloor\frac{A}{s_n}\right\rfloor s_n\right)\right| \leqslant (1 + o(1))b \cdot s_n,$$

which implies that

$$\sup_{\widetilde{I}_{n+1}\setminus\widetilde{\Delta}_{n+1}} |\widetilde{R}_{n+1}(A)| \leqslant \sup_{j} |\widetilde{R}_{n+1}(js_n)| + (1+o(1))b\cdot s_n \to 0.$$

The proof of the fact that the function $\widetilde{R}_{n+1}$ almost surely converges uniformly to zero at the points js_n (as $n\to\infty$) is based on the following observation. The number of points js_n does not exceed h_{n+1}/s_n, and this quantity grows no more rapidly than a polynomial in q_n, while the probability that $|\widetilde{R}_{n+1}(js_n)| > 4\tau_n$ decreases exponentially with respect to q_n, where τ_n is a sequence tending slowly to zero.

Accordingly, we fix some $A = js_n \in \widetilde{I}_{n+1}\setminus\widetilde{\Delta}_{n+1}$. It can be assumed without loss of generality that $A>0$. Let us recall that

$$\widetilde{R}_{n+1}(A) = \int_{\widetilde{I}_n} \widetilde{R}_n(t)\,\widetilde{\nu}_n(dt,A)$$

and $\|\widetilde{R}_n\|_\infty < 1$. Further,

$$\begin{aligned}
\int_{\widetilde{I}_n} \widetilde{R}_n(t)\,\widetilde{\lambda}_n(dt) &= \int_{-h_n}^{h_n} \frac{h_n-|t|}{h_n^2}\widetilde{R}_n(t)\,dt \\
&= \int_{-h_n}^{h_n} \frac{h_n-|t|}{h_n^2}\,dt\biggl(\frac{1}{h_n-|t|}\int_{\mathfrak{I}_n^\circ(t)} f_{(n)}(u+t)\overline{g_{(n)}(u)}\,du\biggr) \\
&= \frac{1}{h_n^2}\int_{-h_n}^{h_n}\overline{g_{(n)}(u)}\,du\int_{-h_n}^{h_n} f_{(n)}(u+t)\,dt = 0.
\end{aligned}$$

We partition the half-open interval $[-1,1)$ of possible values of the function $\widetilde{R}_n$ into half-open intervals $\boldsymbol{i}$ of length τ_n (it is assumed that $\tau_n^{-1}\in\mathbb{N}$). We would like to consider the set $Z_{\boldsymbol{i}} = \widetilde{R}_n^{-1}(\boldsymbol{i})$ and estimate the deviation $|\widetilde{\nu}_n(Z_{\boldsymbol{i}},A) - \widetilde{\lambda}_n(Z_{\boldsymbol{i}})|$. However, the so-defined sets $Z_{\boldsymbol{i}}$ may have a fairly bad structure. Therefore, we define them in another way. Namely, let $\boldsymbol{i}(t)$ be the half-open interval in which $\widetilde{R}_n(t)$ falls. Let $Z_{\boldsymbol{i}(0)}$ contain 0 by definition. Suppose that in moving forward the point $\widetilde{R}_n(t)$ leaves the set $\boldsymbol{i}(0)+[-\tau_n/2,\tau_n/2]$ at the time t_1, that is,

$$t_1 := \inf\{t>0 : \widetilde{R}_n(t)\notin \boldsymbol{i}(0)\}.$$

At the time t_1 the point $\widetilde{R}_n(t_1)$ is in the middle of the interval $\boldsymbol{i}(t_1)$ next to $\boldsymbol{i}(0)$. It will be assumed that $Z_{\boldsymbol{i}(t_1)}$ contains some small interval $(t_1,t_1+\epsilon)$ on which $\widetilde{R}_n(t)\in\boldsymbol{i}(t_1)$. Continuing this procedure, we find the time t_2 of first exit from the $\tau_n/2$-neighbourhood of the interval $\boldsymbol{i}(t_1)$. Then $(t_1,t_2]$ is a connected component of the set $Z_{\boldsymbol{i}(t_1)}$, and the set $Z_{\boldsymbol{i}(t_2)}$ contains a small interval $(t_2,t_2+\epsilon)$, and so on. As a result, we get a partition of $\widetilde{I}_n$ into sets $Z_{\boldsymbol{i}}$ satisfying the following conditions:

1) $|\widetilde{R}_n(t) - c(\boldsymbol{i})| \leqslant \tau_n$ for any $t\in Z_{\boldsymbol{i}}$, where $c(\boldsymbol{i})$ is the centre of $\boldsymbol{i}$;

2) the length of any connected component of $Z_{\boldsymbol{i}}$ is at least τ_n/b.

Suppose that

$$|\widetilde{\nu}_n(Z_{\boldsymbol{i}}, A) - \overline{m}_n^2 \widetilde{\lambda}_n(Z_{\boldsymbol{i}})| \leqslant \tau_n^2 \quad \text{for all } \boldsymbol{i} \tag{7.4}$$

for the given realization of the flow. Then, letting $z(t)$ be the centre of the half-open interval $\boldsymbol{i}$ with $t \in Z_{\boldsymbol{i}}$, we get that

$$|\widetilde{R}_{n+1}(A)| \leqslant \int_{\widetilde{I}_n} |\widetilde{R}_n(t) - z(t)|\, \widetilde{\nu}_n(dt, A) + \sum_{\boldsymbol{i}} c(\boldsymbol{i}) \cdot |\widetilde{\nu}_n(Z_{\boldsymbol{i}}, A) - \overline{m}_n^2 \widetilde{\lambda}_n(Z_{\boldsymbol{i}})| + \int_{\widetilde{I}_n} |\widetilde{R}_n(t) - z(t)|\, \overline{m}_n^2\, \widetilde{\lambda}_n(dt) \leqslant \tau_n + \tau_n^2 \cdot \frac{2}{\tau_n} + \tau_n = 4\tau_n.$$

Here we have used the fact that $\int_{\widetilde{I}_n} \widetilde{R}_n(t)\, \widetilde{\lambda}_n(dt) = 0$. This estimate is what we need. It remains to estimate the probability of realizing the condition (7.4). We consider the random process

$$\eta_{\boldsymbol{i}}(t) = \begin{cases} 1 & \text{if } \theta_n(t, A) \in Z_{\boldsymbol{i}} \text{ and } \Phi_n(t), \Phi_n(t+A) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

and note that

$$\widetilde{\nu}_n(Z_{\boldsymbol{i}}, A) = S_{\boldsymbol{i}} + \mathcal{E}_n, \qquad S_{\boldsymbol{i}} := \frac{1}{h_{n+1} - A} \int_0^{h_{n+1}-A} \eta_{\boldsymbol{i}}(t)\, dt,$$

where $|\mathcal{E}_n| \leqslant 4\overline{q}_n^{-1}$. Then the requirement (7.4) can be replaced by the condition

$$|S_{\boldsymbol{i}} - \overline{m}_n^2 \widetilde{\lambda}_n(Z_{\boldsymbol{i}})| \leqslant \frac{3}{4}\tau_n^2 \quad \text{for all } \boldsymbol{i}, \tag{7.5}$$

since it can be assumed without loss of generality that $16\overline{q}_n^{-1} \leqslant \tau^2$. As above, we have

$$|\overline{\eta}_{\boldsymbol{i}} - \overline{m}_n \widetilde{\lambda}_n(Z_{\boldsymbol{i}})| = |\overline{\eta}_{\boldsymbol{i}} - \overline{m}_n(\lambda_n \times \lambda_n)(D_{n+1}(Z_{\boldsymbol{i}}))| \leqslant e^{c - \omega A/(\kappa_n h_n)}.$$

But by the condition (3) of Theorem 2.5, $A \geqslant \delta_n h_{n+1} = q_n^{1-\chi} h_n \geqslant q_n^{\epsilon} \kappa_n h_n$. Therefore, $A/(\kappa_n h_n) \geqslant q_n^{\epsilon}$, and by properly choosing τ_n we can assume that

$$|\overline{\eta}_{\boldsymbol{i}} - \overline{m}_n \widetilde{\lambda}_n(Z_{\boldsymbol{i}})| \leqslant \frac{1}{4}\tau_n^2.$$

The condition (7.5) (and hence also (7.4)) will hold if we require that

$$|S_{\boldsymbol{i}} - \overline{\eta}_{\boldsymbol{i}}| = \left| \frac{1}{h_{n+1} - A} \int_0^{h_{n+1}-A} \eta_{\boldsymbol{i}}(t)\, dt \right| \leqslant \frac{1}{2}\tau_n^2. \tag{7.6}$$

Let us estimate the probability (7.6). Two cases will be considered for the quantity $a = A/h_n$.

Case 1. $\delta_n q_n \leqslant a < \sqrt{\kappa_n q_n}$. We use ψ-mixing of the process $\eta_{\boldsymbol{i}}$, where $\psi(t) = e^{c_1 - \omega(t-A)/(\kappa_n h_n)}$. We apply Theorem 5.11 to $\eta_{\boldsymbol{i}}$, considering the following values of the parameters: $\boldsymbol{\omega} = \omega/(\kappa_n h_n)$, $\mathbf{c} = c_1 + A/(\kappa_n h_n)$, $\mathbf{h} = h_n$, $\boldsymbol{\rho} = 4$, and $\varepsilon = \tau_n^2/2$.

Then

$$\mathsf{P}\biggl\{|\widetilde{\nu}_n(Z_{\boldsymbol{i}},A)-\overline{\eta}_{\boldsymbol{i}}|>\frac{1}{2}\tau_n^2\biggr\}\leqslant\exp\biggl(-c_1\frac{\tau_n^4}{4}\,\frac{\omega(q_n-a)}{\kappa_n(\ln q_n-\ln\tau_n/32)+2c_1+2\omega a}\biggr)$$
$$\leqslant\exp\biggl(-\frac{q_n-a}{\varkappa_1(q_n)\cdot a}\biggr),$$

where $\varkappa_1(q_n)=o(q_n^\delta)$ for any $\delta>0$.

Case 2. $a\geqslant\sqrt{\kappa_n q_n}$. It can be assumed without loss of generality that $a\ll q_n$ and $a\mid q_n$. We partition $[0,h_{n+1}-A]$ into intervals of length $A/2$ and we fix one of them, $[t_0,t_0+A/2]$. On this interval the process $\eta_{\boldsymbol{i}}$ behaves like a ψ-mixing process with the function $\psi(t)=e^{c_2-\omega t/(\kappa_n h_n)}$. We let $\boldsymbol{\omega}=\omega/(\kappa_n h_n)$, $\mathbf{c}=c_2$, $\mathbf{h}=h_n$, $\boldsymbol{\rho}=4$, and $\varepsilon=\tau_n^2/2$, and we again use Theorem 5.11, obtaining

$$\mathsf{P}\biggl\{|\widetilde{\nu}_n(Z_{\boldsymbol{i}},A)-\overline{\eta}_{\boldsymbol{i}}|>\frac{1}{2}\tau_n^2\biggr\}\leqslant\frac{2q_n}{a}\exp\biggl(-c_2\frac{\tau_n^4}{4}\,\frac{\omega a}{\kappa_n(\ln a-\ln\tau_n/32)+2c_2}\biggr)$$
$$\leqslant\frac{2q_n}{a}\exp\biggl(-\frac{a}{\varkappa_2(a)\cdot\kappa_n}\biggr),$$

where $\varkappa_2(q_n)=o(q_n^\gamma)$ for any $\gamma>0$.

Finally, we consider the following two situations:

(1) $\delta_n q_n\leqslant\sqrt{\kappa_n q_n}$;
(2) $\delta_n q_n>\sqrt{\kappa_n q_n}$.

For (1) it is necessary to consider both case 1 and case 2, and hence the probability has the following bound:

$$\mathsf{P}\biggl\{|\widetilde{\nu}_n(Z_{\boldsymbol{i}},A)-\overline{\eta}_{\boldsymbol{i}}|>\frac{1}{2}\tau_n^2\biggr\}\leqslant\exp\biggl(-\frac{\sqrt{\kappa_n q_n}}{\varkappa_3(q_n)\cdot\kappa_n}\biggr)$$
$$\leqslant\exp\biggl(-\frac{q_n^{(1-(1-\chi-\epsilon))/2}}{\varkappa_3(q_n)}\biggr)=\exp\biggl(-\frac{q_n^{(\chi+\epsilon)/2}}{\varkappa_3(q_n)}\biggr)$$

(where $\varkappa_3(q_n)=o(q_n^\gamma)$ for any $\gamma>0$), which is exponentially small.

In the situation (2) we need consider only case 2, and the corresponding estimate has the form

$$\mathsf{P}\biggl\{|\widetilde{\nu}_n(Z_{\boldsymbol{i}},A)-\overline{\eta}_{\boldsymbol{i}}|>\frac{1}{2}\tau_n^2\biggr\}\leqslant\exp\biggl(-\frac{\delta_n q_n}{\varkappa_4(q_n)\cdot\kappa_n}\biggr)$$
$$\leqslant\exp\biggl(-\frac{q_n^{1-\chi-(1-\chi-\epsilon)}}{\varkappa_4(q_n)}\biggr)=\exp\biggl(-\frac{q_n^{\epsilon}}{\varkappa_4(q_n)}\biggr)$$

(where $\varkappa_4(q_n)=o(q_n^\gamma)$ for any $\gamma>0$), consequently, the desired probability decreases exponentially rapidly also in this case.

To conclude the proof we need an argument analogous to the Borel–Cantelli lemma. Namely, we estimated the probability that the condition $\|\widetilde{R}_{n+1}\|_\infty\leqslant\varepsilon_n$ fails, where ε_n is some sequence going to zero, under the condition that realizations of the processes $\Phi_1,\dots,\Phi_n$ are known. Moreover, the family of events $\{\|\widetilde{R}_{n+1}\|_\infty>\varepsilon_n\}$ is measurable with respect to the σ-algebra generated by $\Phi_1,\dots,\Phi_n$. Finally, the series made up of the bounds of the probabilities of these events converges (even at a rate exponential with respect to $\overline{q}_n$). Therefore, with probability 1 we have $\|\widetilde{R}_{n+1}\|_\infty \leqslant \varepsilon_n$, and hence the estimate $R(A_0) \approx \widetilde{R}_{n+2}(A_0)$ of the correlation function, starting with some n_0.

The proof of the theorem is complete.

Bibliography

[1] A. B. Katok and A. M. Stepin, "Approximations in ergodic theory", *Uspekhi Mat. Nauk* **22**:5 (1967), 81–106; English transl. in *Russian Math. Surveys* **22**:5 (1967).

[2] Ya. G. Sinaĭ (ed.), *Itogi Nauki i Tekhniki*: *Sovrem. Probl. Mat.*: *Fund. Naprav.*, vol. 2, VINITI, Moscow 1985; English transl. in *Encyclopaedia Math. Sci.*, vol. 2 [*Dynamical Systems*, II], Springer–Verlag, Berlin 1989.

[3] I. P. Kornfel'd [Cornfeld], Ya. G. Sinaĭ, and S. V. Fomin, *Ergodic theory*, Nauka, Moscow 1980; English transl., Springer–Verlag, Berlin–Heidelberg–New York 1982.

[4] A. B. Katok and E. A. Sataev, "Standardness of interval exchange automorphisms and flows on surfaces", *Mat. Zametki* **20** (1976), 479–488; English transl. in *Math. Notes* **20** (1976).

[5] E. A. Sataev, "On the number of invariant measures for flows on orientable surfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **39** (1975), 860–878; English transl. in *Math. USSR-Izv.* **9** (1975).

[6] R. V. Chacon, "A geometric construction of measure preserving transformations", *Proc. 5th Berkeley Sympos. Math. Statist. Probab.*, *Univ. Calif.* 1965/66, vol. 2, no. 2, Univ. of California Press, Berkeley 1967, pp. 335–360.

[7] M. Keane, "Interval exchange transformations", *Math. Z.* **141** (1975), 25–31.

[8] V. I. Oseledets, "On the spectrum of ergodic automorphisms", *Dokl. Akad. Nauk SSSR* **168** (1966), 1009–1011; English transl. in *Soviet Math. Dokl.* **7** (1966).

[9] V. I. Oseledets, "An automorphism with simple continuous spectrum without the group property", *Mat. Zametki* **5** (1969), 196–198; English transl. in *Math. Notes* **5** (1969).

[10] M. Keane, "Nonergodic interval exchange transformations", *Israel J. Math.* **26** (1977), 188–196.

[11] P. R. Halmos, *Lectures on ergodic theory*, (Publ. Math. Soc. Japan) Chelsea, NY 1956.

[12] D. S. Ornstein, "On the root problem in ergodic theory", *Proc. 6th Berkeley Sympos. Math. Statist. Probab.*, *Univ. Calif.* 1970, vol. 2, Univ. of California Press, Berkeley 1972, pp. 347–356.

[13] S. Ferenczi, "Systems of finite rank", *Colloq. Math.* **73** (1997), 35–65.

[14] J.-P. Thouvenot, "Some properties and applications of joinings in ergodic theory", *Ergodic Theory and its Connections with Harmonic Analysis* (*K. E. Petersen, ed.*), Cambridge Univ. Press, Cambridge, 1995, pp. 207–235.

[15] J. L. King, "Joining rank and the structure of finite rank mixing transformations", *J. Analyse Math.* **51** (1988), 182–227.

[16] J. L. King and J.-P. Thouvenot, "A canonical structure theorem for finite joining-rank maps", *J. Analyse Math.* **56** (1991), 211–230.

[17] V. V. Ryzhikov, "Joinings and multiple mixing of finite-rank actions", *Funktsional. Anal. i Prilozhen.* **27**:2 (1993), 63–78; English transl. in *Functional Anal. Appl.* **27** (1993).

[18] W. A. Veech, "A criterion for a process to be prime", *Monatsh. Math.* **94** (1982), 335–341.

[19] A. del Junco and D. Rudolph, "On ergodic actions whose self-joinings are graphs", *Ergodic Theory Dynamical Systems* **7** (1987), 531–557.

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