Dynamical systems arising in connection with van der Corput's method and applications to spectral theory

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Littlewood-type polynomials

Question 1. Is it possible to find different real numbers $\omega_1, \dots, \omega_q$ such that the polynomial

$$\mathcal{Q}(t) = q^{-1/2}(e^{i\,t\,\omega_1} + e^{i\,t\,\omega_2} + \ldots + e^{i\,t\,\omega_q})$$

satisfies the *flatness* condition

$$\forall\, t \in \mathbb{R} \qquad \Big| |\mathcal{Q}(t)| - 1 \Big| < \varepsilon$$

for a given $\varepsilon > 0$?

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for a given $\varepsilon > 0$?

The question is open both for real and integer ω_j.

Littlewood-type polynomials

We consider a pair of complementary sums

$$S_0(t) = \cos t\omega_1 + \cdots + \cos t\omega_q$$
$$S_1(t) = \sin t\omega_1 + \cdots + \sin t\omega_q$$

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Then $\mathcal{Q}(t) = q^{-1/2}(S_0(t) + iS_1(t))$, and we ask can it happen that

$$|\mathcal{Q}(t)| \approx 1$$

for any t, or in L^p-sense?

Littlewood's problem on flat polynomials

Question (Littlewood, 1966). Is the following true?
For any $\varepsilon > 0$ there exists a complex polynomial

$$P(z) = \tfrac{1}{\sqrt{n+1}}(c_0 + c_1 z + \cdots + c_n z^n)$$

with unimodular coefficients $|c_j| = 1$ which is ε-*ultraflat* on the unit circle $|z| = 1$ for a given $\varepsilon > 0$ and some $n \geq 1$, that is

$$\forall\, z \in \mathbb{Z}, \quad |z| = 1, \qquad \Big|\, |P(z)| - 1 \,\Big| < \varepsilon$$

Notice that the answer is quite simple in $L^1(S^1)$, but not in $C(S^1)$.

Ultraflat polynomials

Theorem (Kahane, 1980). The answer is "yes" with the accuracy

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Several first polynomials minimizing $\Delta = \big\| \, |\mathcal{P}(t)| - 1 \, \big\|_\infty$

$$\begin{aligned} \deg P = 1 : &\quad \Delta = 1 \\ \deg P = 2 : &\quad \Delta = 2/3 \end{aligned}$$

Understanding the phenomenon of flatness

Observation.

- Further calculation shows tremendous growth of complexity.

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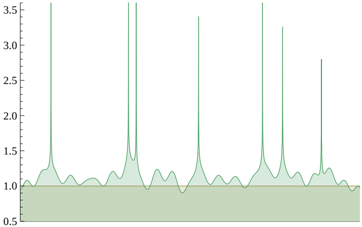
- Further calculation shows tremendous growth of complexity.
- In the talk we consider constructions of different kind. They are in a sense implicit and cause imformation loss.



Dynamical Littlewood's problem

Let us consider the set $\mathcal{G}_{n-1}$ of unimodular polynomials on S^1 of degree $n-1$ as a group according to the multiplication

$$(P \circ Q)(z) = n^{-1/2} \sum_{j=0}^{n-1} a_j b_j z^j,$$

where

$$P(z) = n^{-1/2} \sum_{j=0}^{n-1} a_j z^j, \qquad Q(z) = n^{-1/2} \sum_{j=0}^{n-1} a_j z^j,$$

and $|a_j| = |b_j| = 1$.

Dynamical Littlewood's problem

Question 2. Can we find a family of polynomials

$$P^{(t)}(z) = n^{-1/2} \sum_{j=0}^{n-1} a_j^{(t)} z^j, \qquad a_j^{(t+s)} = a_j^{(t)} \, a_j^{(s)},$$

which are simultaneously flat for $t = t_0, \ldots, t_1$, $t \in \mathbb{Z}$?

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- It extends Littlewood's question on a single uminodular polynomial, and it is also connected to Question 1.
- Question 2 is open in a variety of important cases, e.g.: a family of ultraflat polynomials $P^{(t)}(z)$ on the unit circle.

Physical background

Observation. Our polynomial $\mathcal{P}(t)$ is a particular value of the wave function for the **free quantum particle** driven by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \hat{H}(\boldsymbol{p})\psi,$$

where

$$H(p) = \frac{p^2}{2m}, \qquad \boldsymbol{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}.$$

The function $\omega_j = H(p_j)$ plays the role of the Hamiltonian in the case of discrete bound states.

Physical background

Evolution of the wavefunction: $\hat{\psi}(t, p) = e^{-it\, H(p)/\hbar}\, \hat{\psi}(0, p)$

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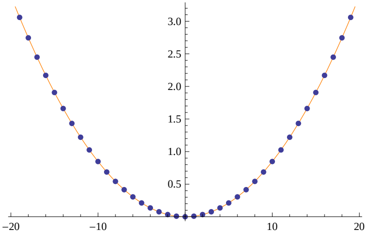
Idea: The structure of a polynomial can be controlled if we consider functions ω_j with moderate local variation, so that we can apply stationary phase method. More exactly, we could require:

- Piecewise C^1
- C^2 with moderate $f'''(x)$



Gaussian sums

All the flat exponential sums are found in this domain?

Gaussian sums

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NO

Example of arithmetic origin: Gaussian sums

$$\left|\sum_{j\in\mathbb{F}_p} e^{2\pi i\, t\frac{j^2}{p}}\right| = \sqrt{p}$$

are flat whenevre p is prime, $t \in \mathbb{Z}$, $(t,p) = 1$.

Estimating exponential sums

Van der Corput's method – Common scheme.
We approximate S as follows

$$S = \sum_{a<k<\beta} \frac{1}{\sqrt{|f''(y_k)|}} e^{2\pi i (f(y_k) - k y_k + 1/8)} + \mathcal{E},$$

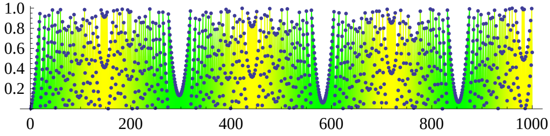
where y_k are the solutions of the equation

$$f'(y_k) = k, \qquad \text{with } k \in \mathbb{Z},$$

and $\alpha = f'(a)$, $\beta = f'(b)$.

Terminology: y_k are the *stationary phase points* for $f(y)$.

Van der Corput's method: Close view



Lemma. Let $f(x)$ be any real function with the derivative decreasing on (a,b). Let $\alpha = f'(a)$, $\beta = f'(b)$. Then

$$\sum_{a<j\le b} e^{2\pi i\, f(j)} = \sum_{\alpha-\eta < k < \beta+\eta} \int_a^b e^{2\pi i\,(f(x)-kx)}\, dx + O(\ln(\beta-\alpha+2)),$$

where $\eta \in (0,1)$ is an arbitrary constant.

Stationary point and phase dynamics

Observation. The polynomial $\mathcal{P}^{(t)}(\theta)$ generates a family of stationary points $y_k(t)$ depending on t,

$$\mathcal{P}^{(t)}(\theta) = q^{-1/2} \sum_{j=0}^{q-1} e^{2\pi i\,(t\,\omega(j)+\theta\,j)}$$

We find $y_k(t)$ solving the equation

$$t\omega'(y_k(t)) = k - \theta,$$

$k \in \mathbb{Z}$, and the evolution of $y_k(t)$ can be expressed in the form of diferential equation.

Stationary point and phase dynamics

Lemma. Let $\omega(y) \in C^2(\mathbb{R})$ with $\omega''(y) > 0$. Set

$$\phi(y) = \omega'(y)$$

and define maps

$$R^t(y) = \phi\left(\frac{\phi^{-1}(y)}{t}\right).$$

Then $\{R^t\}$ is an action of the group $\mathbb{R}_+$ of multiplicative reals, and

$$y_k(t) = R^t\, y_k(0).$$

Stationary point and phase dynamics

Let us introduce $\tau = \log t$.

In the new coordinates $y_k(t)$ follow trajectories of the stationary differential equation

$$\frac{d}{d\tau} y = -\frac{\phi(y)}{\phi'(y)},$$

where $t\dot{y} = \frac{d}{d\tau} y$.

Stationary point and phase dynamics

Let us investigate the essential part in the van der Corput sum:

$$\mathcal{P}^{(t)}(\theta) \approx \mathcal{S}(t,\theta) = \sum_{k\in\mathbb{Z}} \rho_\theta(t, y_k)\, e^{2\pi i\, \Sigma_k(t,\theta)},$$

where $\Sigma_k(t,\theta) = t\omega(y_k) - ky_k$ and $\rho_\theta(t,y) = \hat{R}^t \rho_\theta(0,y)$.

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The generalized Legendre transform $\Sigma_k(t,\theta)$ satisfies the following equations:

$$\dot{\Sigma}_k = \omega,$$

$$t\ddot{\Sigma}_k = ky_k \quad \text{and} \quad \Sigma_{\tau\tau} - \Sigma_\tau = e^\tau \omega_\tau .$$

Examples

Now we consider two examples generating different dynamics

- Quadratic ω case

$$\omega(y) = y^2/2q$$

is the classical Hamiltonian of a free particle on the line

- Hardy–Littlewood case

$$\omega(y) = \frac{q}{\beta^2} e^{\beta y/q}$$

For simplicity we assume that $\theta = 0$.

Quadratic ω case

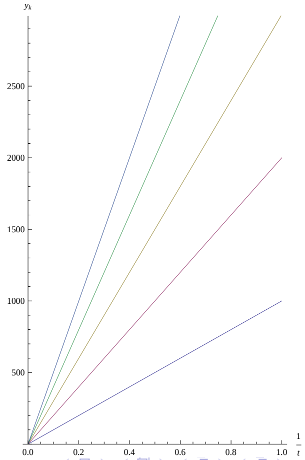
$$\omega'(y) = \frac{y}{q}$$

$$y_k(t) = \frac{q}{t}$$

$$R^t : \; y \to \frac{y}{t}$$

(hyperbolic behaviour)

$$\Sigma_k(t) = \frac{qk^2}{2t}$$



Hardy–Littlewood case

$$\omega'(y) = \frac{1}{\beta} e^{\beta y/q}$$

$$y_k(t) = \frac{q}{\beta} \log \frac{\beta k}{t}$$

$$R^t: \; y \to y - \frac{q}{\beta} \log t$$

(rigid behaviour)

$$\Sigma_k(t) \equiv -k y_k = \xi(t) k + q \cdot \Omega(k)$$

$$\Omega(k) = \frac{1}{\beta} k \log k$$

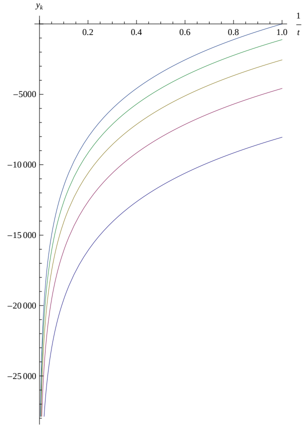


Illustration to the dynamics

Dynamics of R^t

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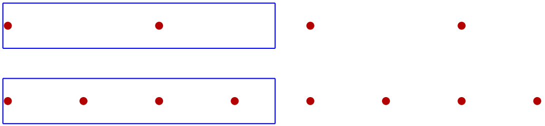

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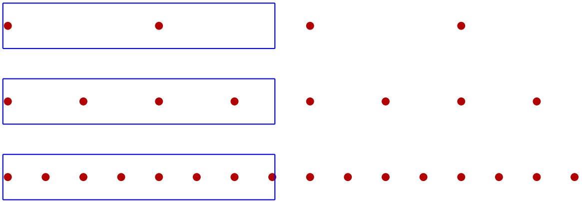

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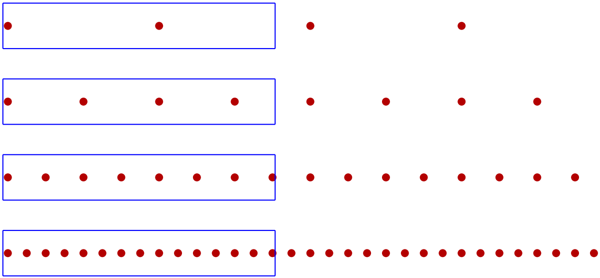

Illustration to the dynamics

Dynamics of S^t

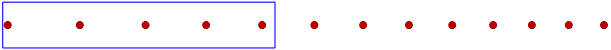

Illustration to the dynamics

Dynamics of S^t

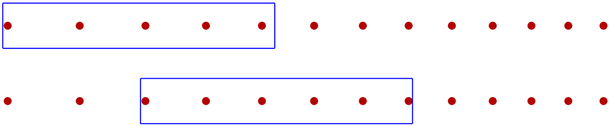

Illustration to the dynamics

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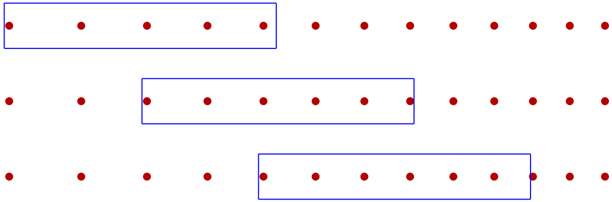
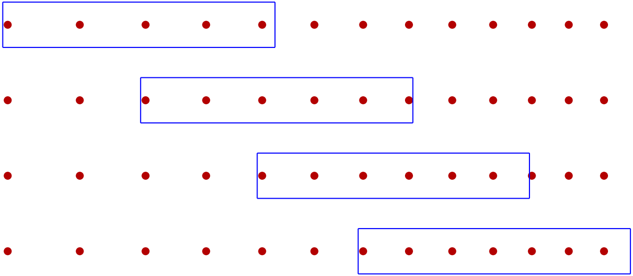


Illustration to the dynamics

Dynamics of S^t



Hardy–Littlewood series

"... We have since discovered that even simplier and more elegant illustrations may be derived from the series

$$\sum e^{\alpha\pi i n + 2\theta\pi i n}.$$

This series behaves, for different values of the parameters α and θ, far more regularly than does the series (1.1). To put the matter roughly, the behaviour of the series *does not, in its most essential features, depend upon the arithmetc nature of* α..."

G.H. Hardy and J.E. Littlewood,
Some problems of diophantine approximation:
A remarkable trigonometric series
Acta Mathematica, 1916

Paradoxical sums and iterated van der Corput's method

Observation. We are interested then in the limit points achieved by the trajectory of $\Sigma_k(t,\theta)$. It can happen that a limit point $\eta(k)$ is approximated by some regular function, and we can repeat the procedure.

Degree reduction:

$$q \quad \longrightarrow \quad 1 \quad \longrightarrow \quad m \quad \dots$$

Question. Is it possible that $m = 1$? Yes

Paradoxical sums and iterated van der Corput's method

Definition.
We say that a sum $\sum_{j=0}^{q} e^{2\pi i\, t\omega(j)}$ is *m-paradoxical* with accuracy ε if the sequence $\Sigma_k(t)$ is ε-close to some C^3-function $\eta(k)$ and

$$\eta'(K_1(t)) - \eta'(K_0(t)) = m \cdot (1+\varepsilon),$$

where $K_0(t) = t\omega'(0)$, $K_0(t) = t\omega'(q)$. We also require regularity conditions, e.g. $\eta^{(3)} = o(\eta'')$ as $\varepsilon \to 0$.

Paradoxical sums: Quadratic ω case

Theorem. Let $m \in \mathbb{N}$ and $\varepsilon > 0$ be given, and let $H = \mathrm{LCM}\{1, \ldots, m\}$. Then for an appropriate q the family of unimodular polynomials

$$P^{(t)}(z) = \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} c_j^t z^j, \qquad c_j = e^{2\pi i\, \omega(j)},$$

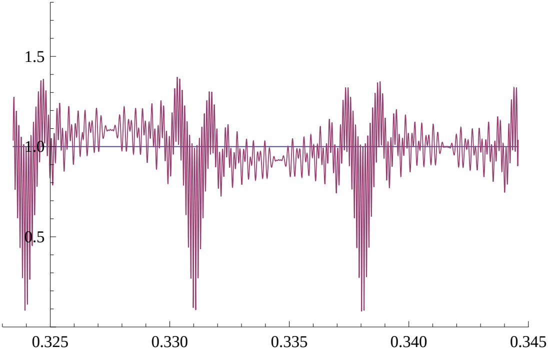
will be flat in L^1-sense for most values of t, where

$$\omega(j) = \omega_0(j) + \Omega(s), \qquad j = s\ell + \varkappa, \quad 0 \le \varkappa < \ell,$$

$$\omega_0(j) = \frac{j^2}{2q}, \qquad \Omega(s) = \frac{s^2}{2H^2}.$$

Paradoxical sums: Quadratic ω case

Sample polynomial $\mathcal{P}^{(t)}(\theta)$



Paradoxical sums: Hardy–Littlewood case

Lemma. Let us consider the following dynamical system given by the shift map on the infinite dimensional torus $\mathbb{T}^\infty$,

$$T\colon\ \mathbb{T}^\infty \to \mathbb{T}^\infty\colon\ x_k \mapsto x_k + \Omega(k).$$

Then according to the weak topology the sequence Ω is the limit point of the trajectory $\{T^q 0\}$, i.e. for any given K_0, K_1 there exists a subsequence q_ℓ such that

$$q_\ell \cdot \Omega(k) \to \Omega(k), \qquad \text{whenever } k \in [K_0, K_1].$$

Proof: Corollary of the Poincaré reccurence theorem.

Paradoxical sums: Hardy–Littlewood case

Theorem. For a given sufficiently large $t_1 > t_0 > 0$ and small parameters $\varepsilon, \beta > 0$ the frequency function

$$\omega(y) = \frac{q}{\beta^2} e^{\beta y / q}$$

generates L^1-flat exponential sums on the segment $[t_0, t_1]$ for a subsequence q_ℓ of positive degree in $\mathbb{Z}$.

Denote $\mathcal{P}(t) = e^{2\pi i \mathcal{A}(t)} + \mathcal{E}(t)$, where $\mathcal{E}(t)$ is L^1-small deviation.

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Question. What is the first reccurence time?

Positive answers: Compact flatness

Theorem. For a given compact set $K \subset \mathbb{R}$ and $\varepsilon > 0$ there exists an L^1-flat exponential sum on K,

$$\Big\| \, |\mathcal{Q}(t)| - 1 \, \Big\|_{L^1(K)} < \varepsilon$$

given by the frequency function

$$\omega(y) = \mu \frac{q}{\beta^2} e^{\beta y/q}$$

Corollary.

- There exists a measure preserving ergodic flow with the Lebesgue spectrum of multiplicity 1.
- On the group $\mathbb{R}$ there exist non-singular Riesz products of Littlewood-type polynomials.

Arithmetic properties of $\Omega(k)$

Question. What are the Diophantine properties of the infinite vector $\Omega(k)$?

Question. What are the Diophantine properties of $\Omega'(k) = \mathrm{const} + \frac{1}{\beta} \log k$?

Van der Corput method is widely used to estimate the asymptotics with respect to $\tilde{t} \to \infty$ of the Riemann zeta function $\zeta(1/2 + i\tilde{t})$. Let us remark that

$$e^{2\pi i (q_\ell \cdot \Omega'(k))} = k^{q_\ell b^{-1}}.$$

Arithmetic properties of $\Omega(k)$

Question. We consider the infinite torus $\mathbb{T}^\infty$ and we are interesting in the limit points of the trajectory $q \cdot \Omega$ as well as its Diophantine properties, where

$$\Omega = (\log 2, \log 3, \log 4, \dots, \log n, \dots)$$

How can we estimate the first return time to zero?

Soliton-like behaviour

Observation. Returning to the physical model we see that our flat sums gives solutions of soliton-like behaviour in the small neighbourhood of a fixed space point θ_0.

Notice that $\omega(y)$ is a small perturbation of $H(y) = y^2/2q$

$$\omega(y) = \frac{q}{\beta^2} + \frac{y}{\beta} + \frac{y^2}{2q} + \beta\frac{y^3}{6q^2} + \dots$$

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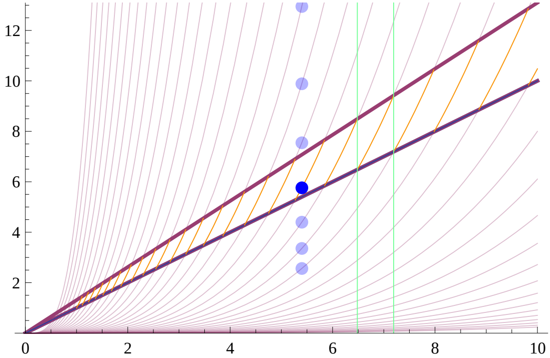
Observation. The effect is not stable with respect to θ.

$\mathcal{Q}(t)$ interpretations

Behaviour of $\mathcal{A}(t)$



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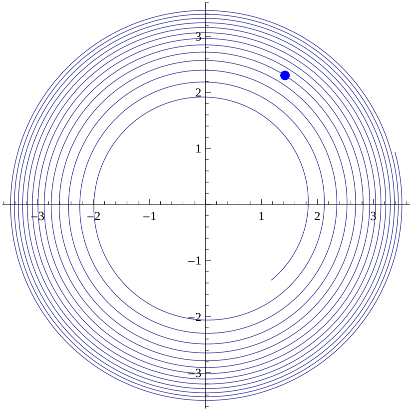
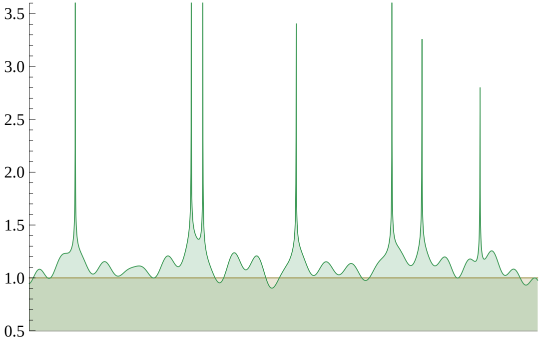


$\mathcal{Q}(t)$ interpretations: Bifurcations

Behaviour of $|P(t)|$



Generalized Riesz products

Problem.
Which Abelian groups $\widehat{G}$ support generalized Riesz producs of Littlewood-type polynomials with coefficients $\{0,1\}$

$$\prod_{n=1}^{\infty} |\mathcal{Q}_n(t)|^2, \qquad \mathcal{Q}_n = q_n^{-1/2} \sum_{\alpha \in A_n} \chi_\alpha(t),$$

converging to an absolutely continuous probability measure on $\widehat{G}$?

Here $\chi_\alpha(t)$ are characters, $A_n \subset G$ and $q_n = \# A_n$.

Salem measures. Combinatorics of divergent series

Definition. A probability measure σ on $[0, 1]$ will be called *Salem measure* if for any $\varepsilon > 0$ the Fourier coefficients

$$\widehat{\sigma}(n) = \int z^n \, d\sigma = O(n^{-1/2+\varepsilon}).$$

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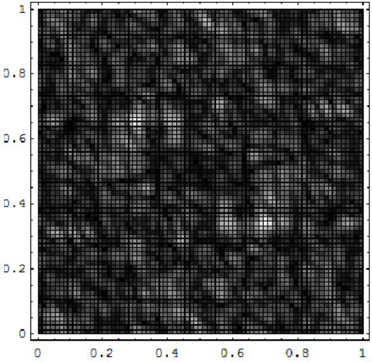
Theorem. Salem spetral measures are observed for flows with Lebesgue spectrum.

Salem measures. Combinatorics of divergent series

How do Salem measures look like? A sample plot on $\mathbb{T}^2$



Salem measures. Combinatorics of divergent series

$$R(n) = O(e^{-\alpha n}) \quad \Rightarrow \quad \text{Spectral type } \sigma \sim \lambda,\ \mathcal{M}(z) = \infty$$

$$R(n) = O(n^{-1/2+\varepsilon}) \quad \Rightarrow \quad ?\left[\begin{array}{l} \sigma \perp \lambda \\ \sigma \sim \lambda \end{array}\right.$$

Problem. How we could distinguis the cases if know the combinatorial properties of the underling dynamical systems?

Salem measures. Combinatorics of divergent series

Thank you!