# Ergodic flows and some problems in analysis

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# Spectral invariants of dynamical systems

Let $T$ be an invertible measure preserving transformation of the standard Lebesgue space $(X, \mathcal{A}, \mu)$, $X = [0, 1]$.
The Koopman operator

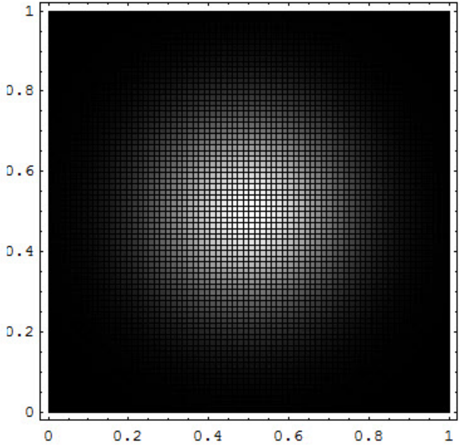
$$\widehat{T}\colon\ L^2(X, \mu) \to L^2(X, \mu)\colon\ f(x) \mapsto f(Tx)$$

Spectral invariants of $T$ are the

- maximal spectral type $\sigma$ on $S^1 = \{z \in \mathbb{C}\colon\ |z| = 1\}$ and the
- multiplicity function $\mathcal{M}(z)\colon\ S^1 \to \mathbb{N} \sqcup \{\infty\}$.

Usually we study $\widehat{T}$ on the space of functions with zero mean.

# Spectral invariants of dynamical systems



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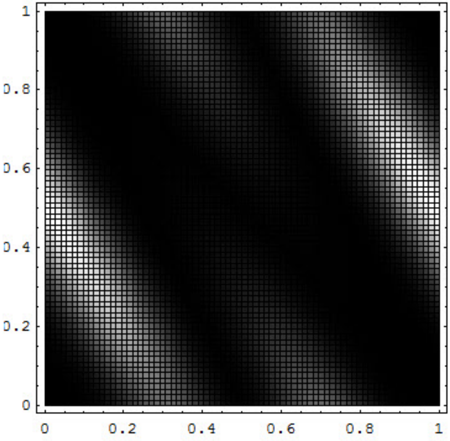

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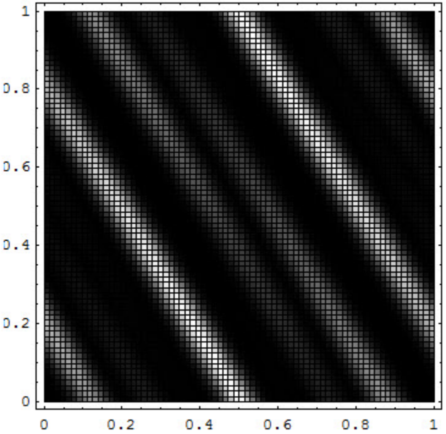

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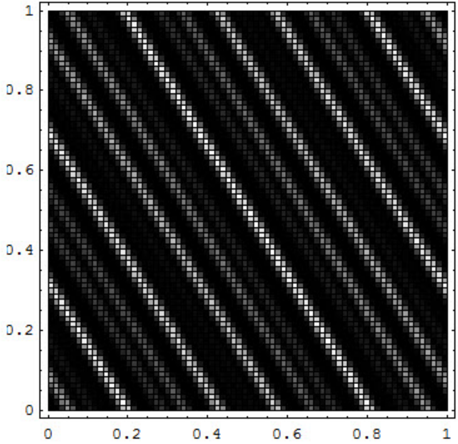

# Spectral invariants of dynamical systems

Let $\{T^t\}_{t\in\mathbb{R}}$ be an ergodic flow on $(X, \mathcal{A}, \mu)$.

We associate with $\{T^t\}$ a unitary representation

$$\widehat{T}^t\colon\ f(x) \mapsto f(Tx)$$

The measure of maximal spectral type for $\{T^t\}$ is a Borel measure on $\widehat{\mathbb{R}} = \mathbb{R}$.

# Spectral invariants of dynamical systems

Examples:

- Bernoulli maps: $\sigma = \lambda$ and the multiplicity $= \infty$
- Transformation with pure point spectrum: spectrum is simple, and $\sigma$ is a distribution on a discrete subgroup in $S^1$ (example: irrational rotation)

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# Singular measures

Riemann (Göttingen, 1854):
Let $f(x)$ be a Riemann integrable function on $[0, 1]$.
Then its Fourier coeffitions

$$\widehat{f}(n) = \int_0^1 e^{-2\pi i nx} f(x)\, dx \to 0, \quad n \to \infty.$$

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# Singular measures

This result is represented in terms of absolutely continuous measures. Let $\sigma$ be a Borel measure on $[0, 1]$. Then

$$\sigma = \sigma_{ac} + \sigma_d + \sigma_s,$$

where

$$d\sigma_{ac} = p(x)d\lambda, \qquad \sigma_d = \sum_{j=1}^{\infty} c_j \delta_{x_j},$$

and

$$\sigma_s \perp \lambda, \qquad [0, 1] = E_1 \sqcup E_2, \quad \lambda(E_1) = \sigma_s(E_2) = 1.$$

# Menshov–Rajchman measures

A measure $\sigma$ is called *Menshov–Rajchman measure* if $\widehat{\sigma}(t) \to 0$ as $t \to \infty$, where

$$\widehat{\sigma}(t) = \int_0^1 e^{-2\pi i\, tx}\, d\sigma(x)$$

Notice that $\widehat{\sigma}_d = \sum_{j=1}^{\infty} c_j \delta_{x_j}$ is an almost periodic function and is non-Menshov–Rajchmann.

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# Menshov–Rajchman measures

**Question.** Which singular measures are Menshov–Rajchman measures?

Menshov (1916):

- For Cantor–Lebeasgue middle-thirds measure $\widehat{\mu}_{\mathrm{CL}}(n) = \widehat{\mu}_{\mathrm{CL}}(3n)$
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# Menshov–Rajchman measures

**Theorem.** There exist Menshov–Rajchman measures with the following rate of Fourier coefficient decay:

$$\widehat{\sigma}(n) = O(n^{-1/2+\varepsilon}).$$

- Wiener and Wintner (1938)
- Schaffer (1939)
- Salem (1943 – 1950)
- Ivashev-Musatov (1956)
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# Symbolic dynamics approach

Let us define *rotation operator* $\rho_\alpha$ on finite words:
If $W = W_{(1)}W_{(2)}$ and the length of the first subword $|W_{(1)}| = \alpha$ then we set $\rho_\alpha(W) = W_{(2)}W_{(1)}$.

Observe that in other terms $\rho_\alpha$ cuts the word $W$ after $\alpha$ positions and then substitutes $W_{(1)}$ and $W_{(2)}$.

This kind of transform is a discrete variation of the well-known *interval exchange map*.

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# Symbolic dynamics approach

Starting from a word $W_0$ define $W_n$ is repeated $q_n$ times, next, each copy is rotated by given value of positions $\alpha_{n,y}$, and the next word in the sequence is given by the formula

$$W_{n+1} = \rho_{\alpha_{n,0}}(W_n)\rho_{\alpha_{n,1}}(W_n)\ldots\rho_{\alpha_{n,q_n-1}}(W_n).$$

For example, if $W_1$ is the word “CAT”, $q_1 = 6$ and $(\rho_{\alpha_{1,0}}, \rho_{\alpha_{1,1}}, \ldots, \rho_{\alpha_{1,q_n-1}}) = (0, 1, 2, 2, 0, 1)$,

# Symbolic dynamics approach

We have

$$\mathrm{CAT} \mapsto \mathrm{CAT.ATC.TCA.TCA.CAT.ATC} = W_2$$

(points "." are used to distinguish groups of symbols). At the next step we rotate the word $W_2$. The following table shows positions of cutting ($\times$)

$$\mathrm{CATATCT}_{\times}\mathrm{CATCACATATC}$$
$$\mathrm{CATA}_{\times}\mathrm{TCTCATCACATATC}$$
$$\mathrm{CATATCTCATC}_{\times}\mathrm{ACATATC}$$

used to create the word

$$W_3 = \mathrm{CATCACATATC} \,|\, \mathrm{CATATCT} \;.\; \mathrm{TCTCATCACATATC} \,|\, \mathrm{CATA} \;.\; \mathrm{ACATATC} \,|\, \mathrm{CATATCTCATC} \ldots$$

# Ergodic construction of Salem measures

**Theorem** (joint work with El H. El Abdalaoui).
There exist measure preserving transformations $T$ having simple spectrum such that the spectral type measure $\sigma$ is purely singular measure with the strictly encreasing distribution function satisfying

$$\widehat{\sigma}(n) = O(n^{-1/2+\varepsilon}) \quad \text{for any } \varepsilon > 0.$$

In particular $\sigma * \sigma \ll \lambda$.

# Ergodic flows

The approach can be extended to the case of flows.

**Theorem.** There exist ergodic flows $T^t$ with simple spectrum such that for a dense set of functions $f$ spectral measures $\sigma_f$ satisfies

$$\widehat{\sigma}_f(n) = O(n^{-1/2+\varepsilon}) \quad \text{for any } \varepsilon > 0.$$

# Ergodic group actions

Let us extend this construction to ergodic actions given by a pair of commuting flows $T^t S^u = S^u T^t$.

**Theorem.** There exist commuting flows $T^t$ and $S^u$ with invariant measure such that for a dense set of functions $f$ spectral measures $\sigma_f$ satisfies

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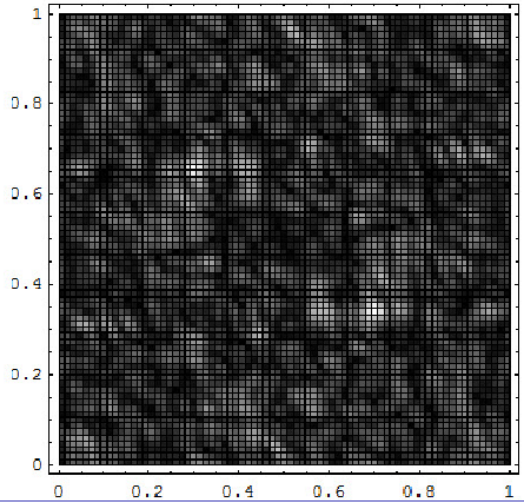

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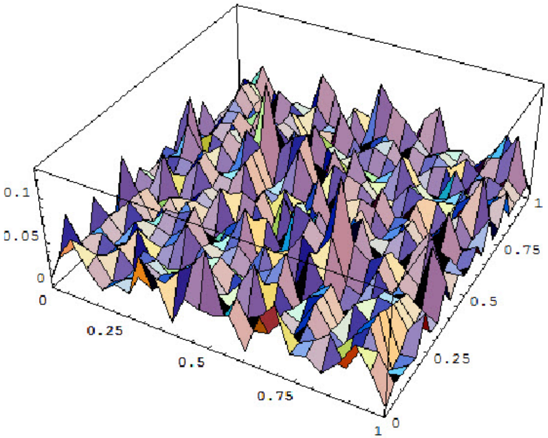

# Thank you!