On Littlewood-type polynomials and applications to spectral theory of dynamical systems

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Polynomials with Littlewood-type coefficient constraints

Definition. A complex polynomial

$$P(z) = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$$

is called *unimodular* if $|a_k| \equiv 1$.

Definition. A polynomial $P(z)$ is called ε-*ultraflat* if

$$\forall z \in S^1 \qquad \big||P(z)| - 1\big| < \varepsilon.$$

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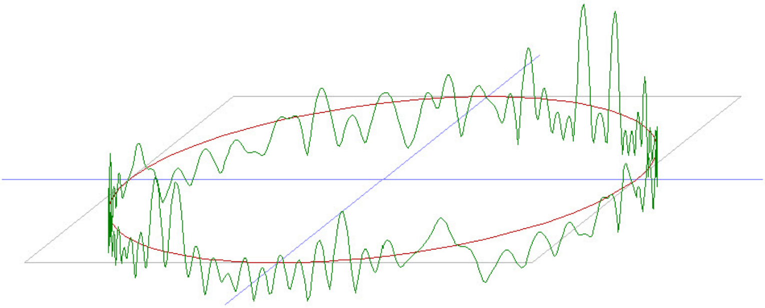
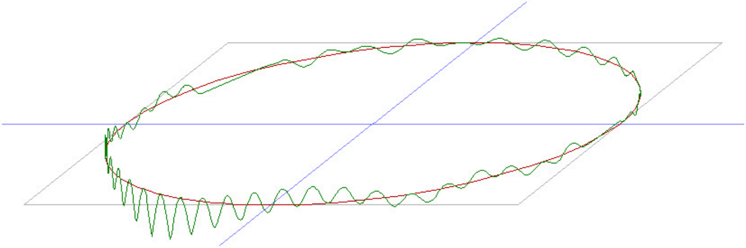
Flatness phenomenon

Polynomial with random coefficients



Flatness phenomenon

Gaussian polynomial $P(z)=\frac{1}{\sqrt{n}}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} a_n z^n,\ a_n=e^{2\pi i\frac{k^2}{2n}}$



Polynomials with Littlewood-type coefficient constraints

$$\mathcal{G}_n = \left\{ P(z) = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k : \ |a_k| \equiv 1 \right\}.$$

$$\mathcal{L}_n = \left\{ P(z) = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k : \ a_k \in \{-1, \ 1\} \right\} \subset \mathcal{G}_n.$$

$$\mathcal{M}_n = \left\{ P(z) = \frac{1}{\sqrt{n}} (z^{\omega_1} + z^{\omega_2} + \ldots + z^{\omega_n}) : \ \omega_j \in \mathbb{Z}, \ \omega_j < \omega_{j+1} \right\}.$$

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Question (Littlewood, 1966). Is the following true?
For any $\varepsilon > 0$ there exists an ε-ultraflat polynomial $P(z) \in \mathcal{G}_n$, $n \geq 1$.

Theorem (Kahane, 1980). The answer is "yes" with the speed of convergence

$$\varepsilon_n = O(n^{-1/17}\sqrt{\ln n}).$$

Question (*open*).
Is it possible to find an ultraflat polynomial in $\mathcal{L}_n$?
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Spectral invariants of dynamical systems

Let T be an invertible measure preserving transformation of the standard Lebesgue space $(X, \mathcal{A}, \mu)$, $X = [0, 1]$.
The Koopman operator

$$\widehat{T}\colon\ L^2(X,\mu) \to L^2(X,\mu)\colon\ f(x) \mapsto f(Tx)$$

Spectral invariants of T are the

- maximal spectral type σ on $S^1 = \{z \in \mathbb{C}\colon\ |z| = 1\}$ and the
- multiplicity function $\mathcal{M}(z)\colon\ S^1 \to \mathbb{N} \sqcup \{\infty\}$.

Usually we study $\widehat{T}$ on the space of functions with zero mean.

Spectral invariants of dynamical systems

Examples:

- Bernoulli maps: $\sigma = \lambda$ and the multiplicity $= \infty$
- Transformation with pure point spectrum: spectrum is simple, and σ is a distribution on a discrete subgroup in S^1 (example: irrational rotation)

Problem (Banach). Is the following true?
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Banach and Kirillov problems

Banach question: *reformulation*
(closer to the original version).

Is the following true? There exists a measure preserving transformation T and an element $\xi \in L^2(X, \mu)$ such that $\widehat{T}^j \xi \perp \widehat{T}^k \xi$ and $\{\widehat{T}^j \xi\}$ generate the entire $L^2(X, \mu)$.

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Answer to Banach problems for $\mathbb{R}$

Answer to Banach (Kirillov) problem is *positive* for group $\mathbb{R}$.

Theorem (P., 2009). There exists a measure preserving flow on a probability space having Lebesgue spectrum of multiplicity one.

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Rank one transformations: Symbolic definition

Any rank one transformation can be described in the following way using the language of symbolic dynamics.

Starting from a word W_{n_0} consider the sequence of words W_n given by

$$W_{n+1} = W_n 1^{s_{n,0}} W_n 1^{s_{n,1}} W_n 1^{s_{n,2}} \dots W_n 1^{s_{n,q_n-1}},$$

where symbol "1" is used to create *spacers* between words, and parameters $s_{n,1}$ are fixed in advance.

Generalized Riesz products

Let us define polynomials

$$P_n(z) = \frac{1}{\sqrt{q_n}} \sum_{y=0}^{q_n-1} z^{\omega_n(y)},$$

where

$$\omega_n(y) = yh_n + s_{n,0} + \ldots + s_{n,y-1}.$$

If $P_n(z)$ are generated by some rank one map, then $P_1(z) \cdots P_n(z)$ always belongs to $\mathcal{M}_{N_n}$.

Generalized Riesz products

The spectral measure σ_f of a function $f \in L^2(X, \mu)$ constant on the levels of a tower with index n_0 is given (up to a constant multiplier) by the infinite product

$$\sigma_f = |\widehat{f}_{(n_0)}|^2 \prod_{n=n_0}^{\infty} |P_n(z)|^2,$$

converging in the weak topology.

Question. Is it possible to construct flat polynomials $P_n(z)$ compatible with some rank one dynamical system?

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Flat exponential sums with coefficients in $\{0, 1\}$

$$\mathcal{M}_q^{\mathbb{R}} = \left\{ \mathcal{P}(t) = \tfrac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i\, t\omega(y)} : \; \omega(y) \in \mathbb{R} \right\}.$$

Theorem. The answer is "yes" in the class $\mathcal{M}_q^{\mathbb{R}}$.
For any $0 < a < b$ and $\varepsilon > 0$ there exists a sum $\mathcal{P}(t) \in \mathcal{M}_q$ which is *compact* ε*-flat* both in $L^1(a, b)$ and $L^2(a, b)$,

$$\Big\| \big|\mathcal{P}(t)\big|_{(a,b)} \big| - 1 \Big\|_1 < \varepsilon,$$

Flat exponential sums with coefficients in $\{0, 1\}$

and the sums $\mathcal{P}(t)$ are given by the formula

$$\mathcal{P}(t) = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i \, t\omega(y)},$$

where

$$\omega(y) = m\frac{q}{\beta^2} e^{\beta y/q},$$

with appropriate choice of $m > 0$, $\beta^{-1} \in \mathbb{N}$ and q ranging over a set $\mathcal{Q}_{\varepsilon,a,b}(\beta, \varepsilon, m)$ of positive density in $\mathbb{Z}$.

Exponential staircase flow

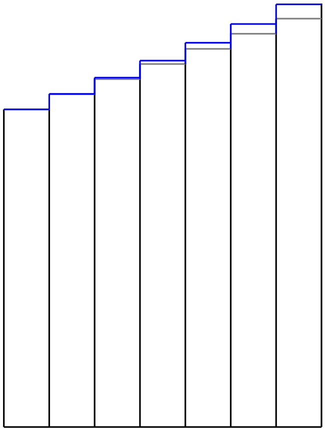
We construct a rank one flow with the following parameters:

- q_n is the number of subcolumns
- spacers $s_{n,y} = \omega_n(y+1) - \omega_n(y) - h_n$
- $\omega_n(y) = \mu_n \dfrac{q_n}{\beta_n^2} e^{\beta_n y/q_n}, \qquad h_n = \dfrac{\mu_n}{\beta_n}$

$\mu_n \to \infty$ (*slowest*), $\beta_n \to 0$, $q_n \to \infty$ (*fastest*).

Theorem. With certain choice of parameters μ_n, β_n and q_n the rank one flow given by the exponential staircase construction has Lebesgue spectral type.

Exponential staircase flow generated by $\omega(y) = \frac{q}{\beta} e^{\beta y/q}$



"…though careful examination had shown that the height of the steps steadily decreased with the rising gravity. The stair had apparently been designed so that the effort required to climb it was more or less constant at every point in its long curving sweep…"

Arthur C. Clarke, *Rendezvous with Rama*

Exponential sums: Van der Corput's method

Van der Corput's method – The concept.
We can estimate S as follows:

$$S = \sum_{a<k<\beta} \frac{1}{\sqrt{|f''(y_k)|}} e^{2\pi i (f(y_k) - k y_k + 1/8)} + \mathcal{E},$$

where y_k are solutions of the equation

$$f'(y_k) = k, \qquad \text{where } k \in \mathbb{Z},$$

and $\alpha = f'(a)$, $\beta = f'(b)$.

Points y_k are called *stationary phases* for the function $f(y)$.

Physical background: Schrödinger equation

Let $H(p) = \frac{p^2}{2m}$, and consider equation for a free particle

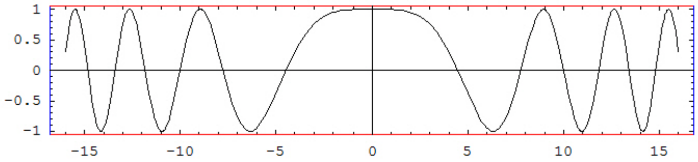
$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2},$$

de Broigle wave

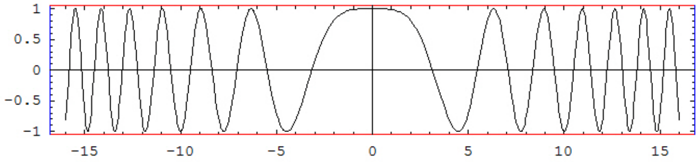
$$\psi(x,t) = \exp\left(\frac{p_0 x - \omega_p t}{\hbar}\right), \qquad \omega_p = H(p),$$

and the group velocity p_0 given by $p_0 - H'(p) = 0$.

Fundamental solution on $\mathbb{R}$



- Graphics -



Free quantum particle on a compact space

Remark that the sum $\mathcal{P}(t)$ is connected to a quantum dynamical system given on $\mathbb{T}$ by equation

$$i\frac{\partial}{\partial t}\psi = H\psi,$$

with

$$H = \omega\left(-i\frac{\partial}{\partial x}\right),$$

and $\mathcal{P} = \hat{\psi}$.

Stationary phase dynamics

Our sum $\mathcal{P}(t)$ generates a family of stationary phases $y_k(t)$ depending on t,

$$\mathcal{P}(t) = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i\, t\omega(y)},$$

given by the equation

$$t\omega'(y_k(t)) = k,$$

and the law of evolution for $y_k(t)$ in some cases is expressed by a dynamical system $\dot{y} = v(t, y)$.

Stationary phase dynamics: Example

Quadratic function $\omega(y) = H_0(y) = \frac{y^2}{2q}$ generates $y_k(t)$ as follows:

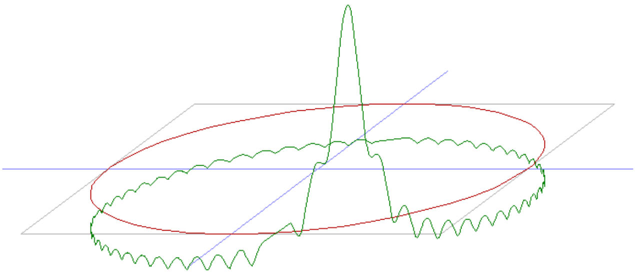
$$t \cdot H_0'(y_k) = k, \quad t \cdot \frac{y_k}{q} = k, \quad y_k(t) = \frac{kq}{t},$$

and the dynamical system induced by H_0 acts on $\mathbb{R}$ as follows:

$$R^t \colon \; x \mapsto \frac{x}{t}, \qquad y_k(t) = R^t y_k(0).$$

Notice that R is an action of the multiplicative group $\mathbb{R}_+$.

Quantum chaos phenomenon



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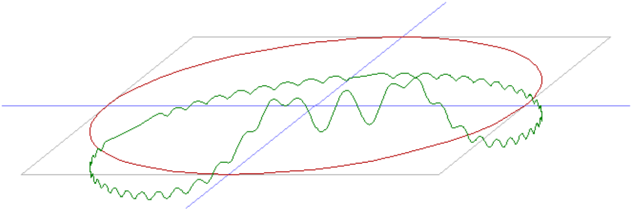

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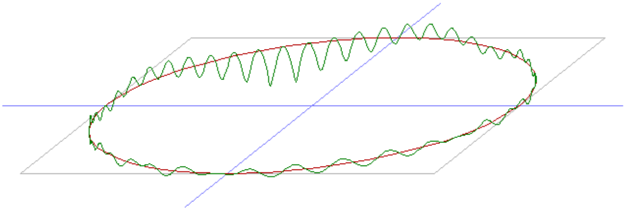

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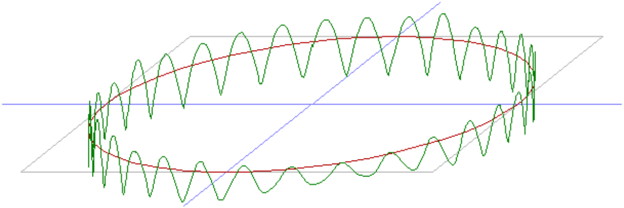

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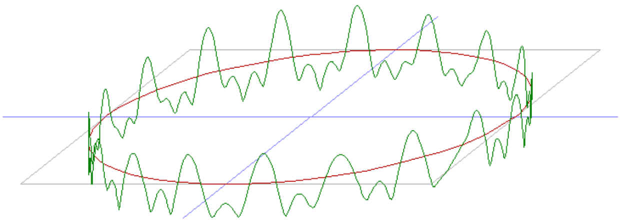

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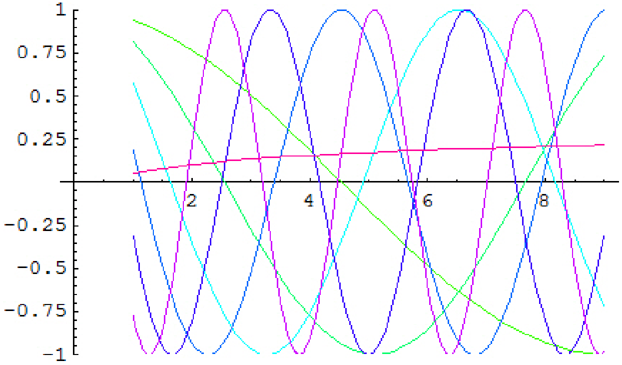
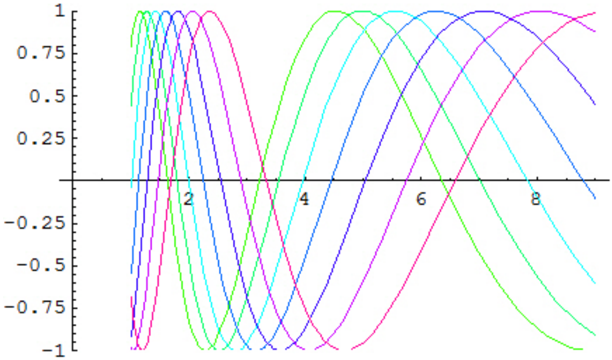


Quantum chaos phenomenon: Rigid case



Stationary phase dynamics: Idea

Ways of constructing flat $\mathcal{P}(t)$:

- Searching for special unstable cases of arithmetic nature. Example: $2mH_0(y) = my^2/q$, $2m \in 2\mathbb{Z}$ and q is prime.
- Controlling the dynamics of stationary phases $y_k(t)$.

Idea:

(a) The set $\{t\omega(y_k)\}$ is generally chaotic (e.g. for H_0). Could it be constant for some special choice of $\omega(y)$?
or

(b) Is it possible to control the distances: $y_{k+1}(t) - y_k(t) = \mathrm{const}$?

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Stationary phase dynamics: Calculation

Let us suppose that $\frac{d}{dt}(t\omega(y_k)) = 0$, then

$$\omega(y_k) + t\omega'(y_k)\dot{y}_k = 0.$$

Now differentiating the equation $t\omega'(y_k) = k$ we get

$$\omega'(y_k) + t\omega''(y_k)\dot{y}_k = 0,$$

therefore

$$\frac{\partial}{\partial y}\frac{\omega'}{\omega} = 0, \qquad \frac{\omega'}{\omega} = \beta q^{-1} = \mathrm{const},$$

and $\omega(y) = \omega_0 e^{\beta y/q}$.

Stationary phase dynamics for exponential $\omega(y)$

Observe that $\omega(y)$ has expansion (with small parameter β)

$$\omega(y) = \frac{q}{\beta^2} + \frac{y}{\beta} + \frac{y^2}{2q} + \beta\frac{y^3}{6q^2} + \ldots$$

Let us associate an $\mathbb{R}_+$-action to our $\omega(y)$. Solving equation

$$t \cdot \omega'(y) = k = \mathrm{const}$$

we have

$$t \cdot \frac{1}{\beta} e^{\beta y/q} = k, \qquad y_k(t) = \frac{q}{\beta} \log \frac{\beta k}{t},$$

Stationary phase dynamics for exponential $\omega(y)$

$$y(t) = y(0) + \frac{q}{\beta} \log t^{-1},$$

and the dynamical system $S^t\colon\ y(0) \mapsto y(t)$,

$$S^t\colon\ x \mapsto x + \frac{q}{\beta} \log t^{-1}$$

acts by translations of the line $\mathbb{R}$.

Dynamical system observation: S^t is much less “chaotic” than R^t.

- R^2 acts on 1-periodic functions as *hyperbolic map*
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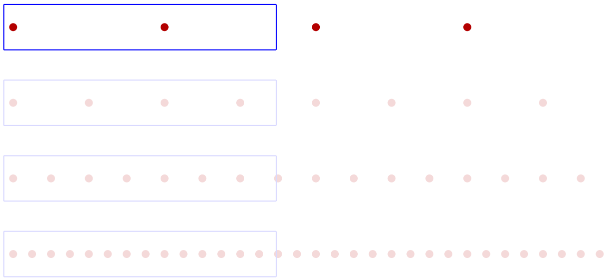
Illustration to the dynamics

Dynamics of R^t



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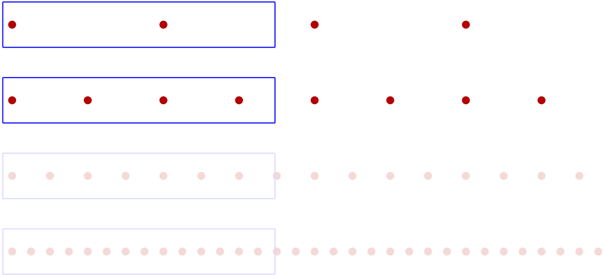

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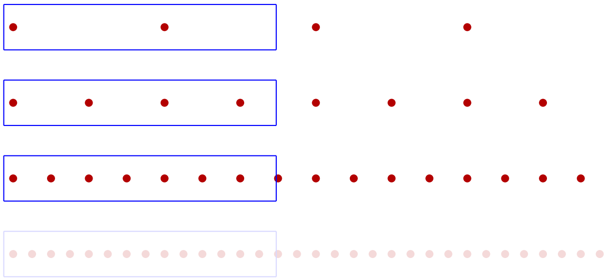

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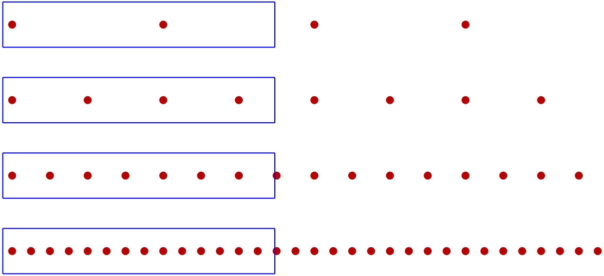
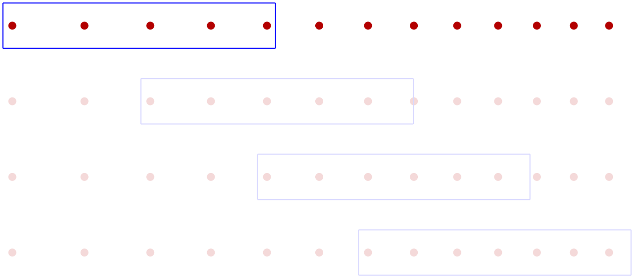


Illustration to the dynamics

Dynamics of S^t



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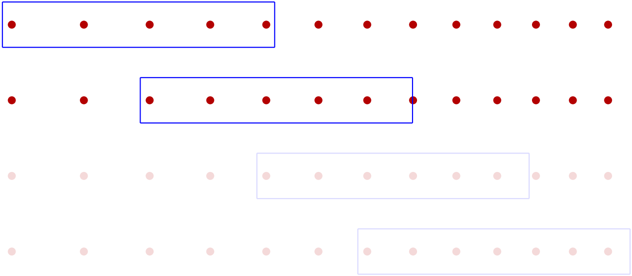

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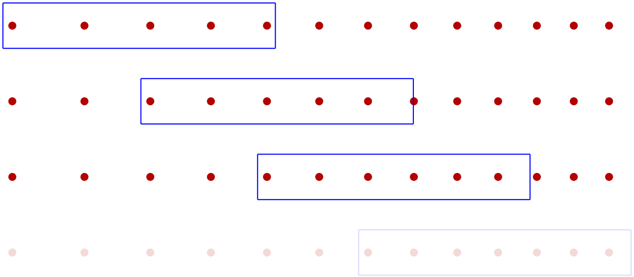

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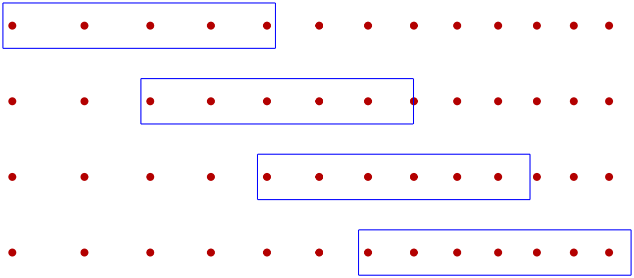

Outline of the proof: Calculation of $y_k(t)$

We see that

$$y_k(t) = \frac{q}{\beta} \log \frac{\beta k}{t},$$

where the index $k \in \mathbb{Z}$ ranges over the interval $(K_0(t), K_1(t))$,

$$K_0(t) = \frac{t}{\beta}, \qquad K_1(t) = \frac{t}{\beta} e^{\beta},$$

$$K_1(t) - K_0(t) \sim t, \qquad \beta \to 0, \quad t \to \infty.$$

Outline of the proof: Van der Corput's method

Applying van der Corput approach we have for $t \to \infty$

$$\mathcal{P}(t) = \frac{1}{\sqrt{t}} \sum_{K_0(t)<k<K_1(t)} e^{2\pi i(t\omega(y_k)-ky_k+1/8)} + \mathcal{E}_1(t).$$

Let us calculate the resulting phase function (minus $1/8$)

$$t\omega(y_k) - ky_k \equiv -ky_k \pmod 1,$$

since

$$t\omega(y_k) = \frac{q}{\beta} \cdot t\omega'(y_k) = \frac{q}{\beta} \cdot k \in \mathbb{Z},$$

if we require that $\beta^{-1} \in \mathbb{Z}$.

Applying van der Corput's method twice

Continuing calculation of the phase function we have

$$-ky_k = -k \cdot \frac{q}{\beta} \log \frac{\beta k}{t} = x(t)k - q \cdot \Omega(k),$$

where

$$x(t) = \frac{q}{\beta} \log \frac{t}{\beta}$$

do not depend on k, and

$$\Omega(k) = \frac{1}{\beta} k \log k$$

do not depend on t.

Applying van der Corput's method twice

Theorem (Poincaré reccurence theorem). Given $\varepsilon > 0$ for a sequence of q of positive density

$$-q \cdot \Omega(k) \approx_{\varepsilon} \Omega(k),$$

for the fixed finite set of $k \in (K_0, K_1) \cap \mathbb{Z}$.

Here we apply the reccurence theorem to the torus shift on $\mathbb{T}^{[t]}$

$$T\colon\ v \mapsto v + \Omega.$$

Applying van der Corput's method twice

It can be easily seen that

$$\Omega'(K_1(t)) - \Omega'(K_0(t)) = 1 + o(1),$$

as $\beta \to 0$ and $t \to \infty$, hence, applying again van der Corput estimate we have

$$\mathcal{P}(t) \approx \frac{1}{\sqrt{t}} \sum_{K_0(t)<k<K_1(t)} e^{2\pi i\,(-ky_k+1/8)} = e^{2\pi i\,\mathcal{A}(t)} + \mathcal{E}_2(t),$$

where $\mathcal{E}_2(t)$ is L^p-small error term for $p = 1$ and $p = 2$.

Scheme of the approach

Exponential sum with coeffitients $\{0, 1\}$

$\rightarrow$

Van der Corput's method (1), reduction: degree q to degree t

$\rightarrow$

Quantum free particle on $\mathbb{T}$

$\rightarrow$

Dynamical system: $\mathbb{R}_+$-action induced by the Hamiltonian $\omega(y)$

$\rightarrow$

Dynamical system on $\mathbb{T}^{[t]}$ given by a torus shift

$\rightarrow$

Van der Corput's method (2), reduction: degree t to degree 1

Thank you!