Research Statement

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1 Transitions of minimum functions and shock waves

Let $F(\mathrm{u}, \lambda)$ be a smooth family of functions of u depending on a parameter $\lambda \in \mathbb{R}^{d+1}$. Then the function

$$\varphi(\lambda) = \min_{\mathrm{u}} F(\mathrm{u}, \lambda)$$

of the parameter is called the *minimum function* of the family F. In general, minimum functions are not smooth. In some applications

$$\lambda = (\mathrm{x}, t) \in \mathbb{R}^{d+1}$$

is a point of the space-time — in this case we say that the discontinuities of F form the *world shock*, and the cases $d = 2, 3$ are the most interesting physically. At a fixed t_0 the discontinuities of F form the *instant shock* which is a section of the world shock with the isochrone $t = t_0$. As t_0 varies the instant shock experiences transitions.

All such possible transitions when F is a typical family are shown in Fig. 1 ($d = 2$), Fig. 2 ($d = 3$)

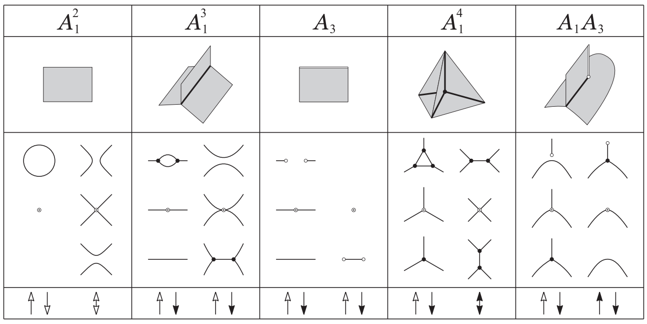

Figure 1: Singularities of world shocks and transitions of instant shocks in plane

and classified in [1] where the obtained classification is applied to inviscid irrotational solutions to the forced Burgers equation:

$$\begin{cases} \mathrm{v}_t + (\mathrm{v} \cdot \nabla)\,\mathrm{v} = -\nabla U + \nu\,\Delta\,\mathrm{v} \\ \mathrm{v} = \nabla\psi \\ \psi(\mathrm{x}, 0) = \varphi_0(\mathrm{x}) \end{cases}.$$

Here $\mathrm{x} \in \mathbb{R}^d$ is a point of the medium, $\mathrm{v}(\mathrm{x}, t)$ is the velocity at the point x at the time t, $\nu > 0$

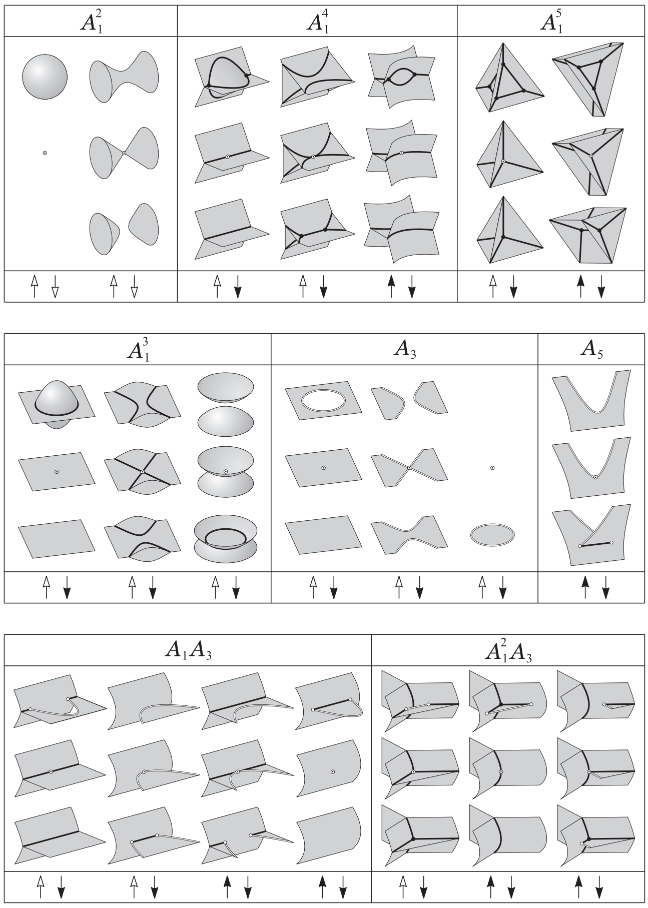

Figure 2: Transitions of instant shocks in space

is the viscosity of the medium, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$ is the usual ∇-operator in $\mathbb{R}^d$, and $\Delta = \nabla \cdot \nabla$ is the Laplacian. The force potential $U(\mathrm{x}, t)$ and the initial condition ψ_0 are assumed to be smooth.

Let φ be the limit solution as the viscosity vanishes. It has the well-known minimum representation:

$$\varphi(\mathrm{x}, t) = \lim_{\nu \to 0} \psi(\mathrm{x}, t) = \min_{\mathrm{u}} F(\mathrm{u}, \mathrm{x}, t)$$

and is called an *inviscid* solution. For example, in the unforced case $U \equiv 0$

$$F(\mathrm{u}, \mathrm{x}, t) = \varphi_0(\mathrm{x} - \mathrm{u}t) + t|\mathrm{u}|^2/2.$$

Therefore we can apply our classification of singularities and transitions of shocks to such limit solutions. But not all transitions from Figures 1 and

2 are realized by inviscid irrotational solutions to the Burgers equation — the realizable transitions satisfy the following topological restriction:

A local shock after a transition is contractible (homotopically equivalent to a point). For example, the triangle from shocks is homotopically a circle but not a point, hence it cannot appear. The realizable transitions are shown in Figures 1 and 2 by black arrows.

Inviscid solutions to the Burgers equation explain the formation of clusters in the Universe. Let us consider two-dimensional case shown in Fig. 1. The linear density of particles is positive on the shock. However, the natural assumption that triple nodes are clusters of particles having positive mass is not correct — it was discovered in [2].

Namely, there are attracting nodes (with three obtuse angles) and neutral nodes (with an acute angle). The relative velocities of particles (in the frame connected with the node) are shown in Fig. 3. Therefore, a cluster appears and starts growing when a neutral node becomes attracting (Fig. 4, left). If the node becomes neutral again the cluster leaves it and stops growing (Fig. 4, right).

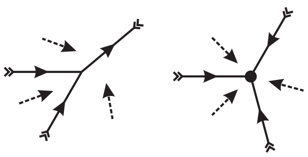

Figure 3: Motion of particles around neutral and attracting nodes

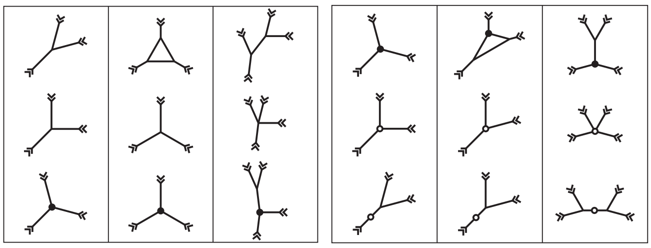

Figure 4: Growing (black dots) and stable (white dots) clusters

These observations are implied by the following results obtained in [2]. It turns out that a minimum function has a gradient and the ordinary differential equation $\dot{x} = \nabla\varphi$ defines correctly the limit motion of particles.

Research plans. To study singularities and transitions of shocks in gas dynamics. It is a more complicated problem because the system of partial differential equations has more than one unknown function. Singularities of one-dimensional many-valued barotropic gas flows are investigated in [3].

References to Section 1

[1] I. A. Bogaevskii. Perestroikas of shocks and singularities of minimum functions. *Physica D: Nonlinear Phenomena*, 173(1-2):1–28, 2002.

[2] I. A. Bogaevskii. Discontinuous gradient differential equations and trajectories in calculus of variations. *Sb. Math.*, 197(12):1723–1751, 2006.

[3] I. A. Bogaevskii and D. V. Tunitsky. Singularities of multivalued solutions of quasilinear hyperbolic systems. *Proc. Steklov Inst. Math.*, 308:67–78, 2020.

2 Fronts of singular Legendrian submanifolds

Singular Legendrian submanifolds appear in various areas of Mathematics. For example, the open swallowtails and open Whitney umbrellas are well-known in the obstacle problem, and in 1990 V. I. Arnold described a Legendrian submanifold with non-analytic singularities appearing in short-wave approximations in some problems of Mathematical Physics. This submanifold is the closure of the 1-jet graph:

$$u = x_1^2 \log x_2, \quad p_k = \partial_{x_k} u, \quad k = 1, \dots, n, \tag{1}$$

its dimension $n \geq 2$.

A universal technique for reducing the fronts of singular Legendrian submanifolds to normal forms is developed in [4]. The Lagrangian counterpart of this technique is developed in [5] and makes possible to compute normal forms of caustics.

The fronts, their transitions, and caustics of the Legendrian submanifold (1) are investigated in [6] and [7].

References to Section 2

[4] I. A. Bogaevskii. New singularities and perestroikas of fronts of linear waves. *Moscow Math. J.*, 3(3):807–821, 2003.

[5] I. A. Bogaevskii and G. Ishikawa. Lagrange mappings of the first open Whitney umbrella. *Pacific J. Math.*, 203(1):115–138, 2002.

[6] I. A. Bogaevskii. Front transitions of interior scattering. *Doklady Mathematics*, 83(1):26–29, 2011.

[7] I. A. Bogaevskii. Caustics of interior scattering. *Proc. Steklov Inst. Math.*, 267:1–7, 2009.

3 Graphene, the Dirac equation, and sub-Lorentzian structures

It turns out that fronts of the singular Legendrian submanifold (1) appear as the graphs of phases in semiclassical approximations of solutions to the massless Dirac equation and describe the motion of electrons and holes in graphene.

In [8] we consider the fundamental solution to the stationary two-dimensional massless Dirac equation

$$[x\sigma_0 + \sigma_x \hat{p}_x + \sigma_y \hat{p}_y]\,\psi(x,y) = h^{\frac{3}{2}}\,\delta(x+a,y)\,\mathrm{w} \quad (2)$$

where $a > 0$; $(x,y) \in \mathbb{R}^2$ is a point in the plane;

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{C}^2, \quad \mathrm{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{C}^2$$

are an unknown spinor field and a known spinor respectively;

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

are Pauli matrices; δ is the Dirac delta-function; the coefficient x before the matrix σ_0 is the potential energy of a quasiparticle in a constant electric field; $\hat{p}_x = -ih\,\partial_x$ and $\hat{p}_y = -ih\,\partial_y$ are momentum operators; and $h > 0$ is a small real parameter.

If the limit-absorption principle is satisfied, then the solution to (2) is unique and describes the stationary flow of quasiparticles in graphene emitted with zero energy from a source located at the point $x = -a < 0$, $y = 0$. There is a constant electric field parallel to the axis x, and the potential energy in this field of a quasiparticle is $U(x,y) = x$. Quasiparticles in graphene (electrons for $U < 0$ and holes for $U > 0$) are fermions with a zero effective mass and the Fermi velocity $v_{\mathrm{F}} \approx 10^6$ mps (effective speed of light in graphene) but we set $v_{\mathrm{F}} = 1$.

The source described by (2) emits

$$a\frac{|w_1|^2 + |w_2|^2}{2}$$

electrons per the unit time. The propagation of rays is shown in Fig. 5. Rays leaving the source in all directions except one remain in the electron region. Their envelope has an infinite curvature at the point O and is a caustic at which the electron density tends to infinity as $h \to 0+$. The Lagrangian surface defined by these rays is neither smooth nor analytic: its normal form found by V. I. Arnold in 1990 contains logarithms.

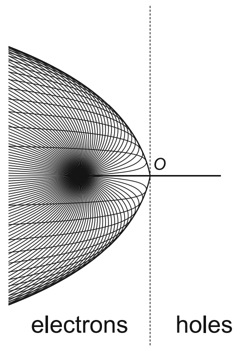

Figure 5: Propagation of rays from a point source in a constant electric field

In [8] we find the explicit asymptotic of the solution to (1) as $h \to 0+$ for $x > 0$:

$$\psi(x,y) \approx \frac{i(w_1+w_2)}{2\sqrt{2\pi\mathcal{A}}}\, e^{-\frac{y^2}{2h\mathcal{A}}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathcal{A} = \pi - i\log\frac{x}{a}.$$

From the physical point of view, this is the asymptotic of the holes formed from the electrons emitted by the source. According to our results, a part of the electrons proportional to $\sqrt{h}$ becomes holes. These holes are localized along the half-line $x > 0$, $y = 0$ with a characteristic size $\sqrt{h}$ and their density decreases rapidly as $y \to \infty$. Our asymptotic works in the hole region only. At the level of geometric optics, it is explained by a special ray that breaks into the hole region and passes far from other rays that remain in the electron region.

Surprisingly, rays in graphene located in electromagnetic field are null-geodesics of a sub-Lorentzian structure defined by the field. Besides, the graphs of the phases of semiclassical approximations of solutions to the massless Dirac equation are fronts of this sub-Lorentzian structure.

Sub-Lorentzian structures defined by electromagnetic fields in graphene are families of Lorentzian metrics on distributions of three-dimensional planes in $\mathbb{R}^4$. Such sub-Lorentzian structures and their fronts are studied in [9] and [10].

Research plans. To compute the semiclassical asymptotic of the analogous transition electrons to holes in electromagnetic field. To construct the leading centre theory for quasiparticles of graphene in a strong electromagnetic field.

References to Section 3

[8] I. A. Bogaevskii. Fundamental solution of the stationary Dirac equation with a linear potential. *Theoret. and Math. Phys.*, 205(3):1547–1563, 2020.

[9] I. A. Bogaevskii. Sub-Lorentzian structures in $\mathbb{R}^4$: Left-invariance and conformal normal forms. *Journal of Dynamical and Control Systems*, 24(3):371–389, 2018.

[10] I. A. Bogaevskii. Fronts of control-affine systems in $\mathbb{R}^3$. *Journal of Singularities*, 21:15–29, 2020.

4 Sub-Riemannian geometry and optimal control

A sub-Riemannian structure is a family of Euclidean metrics on a distribution of k-dimensional planes in $\mathbb{R}^n$. (If $k = n$ we just get a Riemannian structure.) A sub-Riemannian structure naturally defines a control system in $\mathbb{R}^n$ — at any point the admissible velocities form the unit k-dimensional disk. Therefore the sub-Riemannian geometry studies control systems of this special kind.

A curve is called *integral* if it is tangent to the distribution at any its point. Any segment of an integral curve has a length. The *distance* between two points is the minimal length of an integral curve connecting them. (If there are no such curves the distance is not defined.)

An integral curve is called *optimal* if the length of any its segment is equal to the distance between its ends. The *sub-Riemannian sphere* of a radius $R > 0$ with a centre $a \in \mathbb{R}^n$ is the set of the ends of all optimal curves of the length R beginning at a.

An integral curve is called *locally optimal* if it can be covered by overlapping optimal curves. The *sub-Riemannian front* of a radius $R > 0$ with a centre $a \in \mathbb{R}^n$ is the set of the ends of all locally optimal curves of the length R beginning at a. (The sub-Riemannian sphere is a subset of the front.)

Let us fix a point $a \in \mathbb{R}^n$. In the Riemannian case $k = n$ for a small $R > 0$ the sphere and front coincide and are diffeomorphic to the standard $(n-1)$-dimensional sphere. In the sub-Riemannian case the situation can be quite different: the well-known first interesting case of a contact distribution is shown in Fig. 6. In this case $n = 3$, $k = 2$,

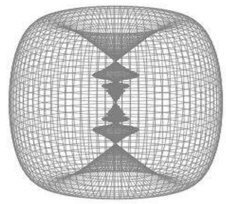

Figure 6: Sub-Riemannian front (contact distribution)

$dz = y\,dx$ is the distribution, $dx^2 + dy^2$ is the metric, and the sub-Riemannian sphere and front are given by explicit parametric formulas in elementary functions. Nevertheless, the front is not analytic at the origin $x = y = z = 0$.

For other sub-Riemannian structures, being interesting for applications, the singularities of their spheres and fronts are more complicated. Sometimes the fronts are given by explicit parametric formulas which contain special functions, but these formulas do not help to understand what happens in neighbourhoods of singular points of the fronts.

The general problem is to study singularities of sub-Riemannian spheres and fronts of small radii. In [11] and [12] we compute the quasihomogeneous tangent cones to the sub-Riemannian fronts at non-analytic points of the spheres for the Martinet distribution $dz = y^2\,dx/2$ in $\mathbb{R}^3$ and the Engel distribution in $\mathbb{R}^4$. In both cases we consider the flat metric $dx^2 + dy^2$.

Research plans. Presumably the method developed in [11], [12] can be applied in other cases as well. The first difficult case is the Martinet distribution with a non-flat metric.

References to Section 4

[11] I. A. Bogaevskii. Asymptotics of the sphere and front of a flat sub-Riemannian structure on the Martinet distribution. *Sb. Math.*, 213(5):624–640, 2022.

[12] I. A. Bogaevskii. Asymptotics of the sphere and front of a flat sub-Riemannian structure on the Engel distribution. *Proc. Steklov Inst. Math.*, 2022. Accepted.