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ERGODIC JOININGS OF GL$(n,\mathbb{Z})$-ACTION ON n-TORUS

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ABSTRACT. In this paper a classification of ergodic self-joinings of GL$(n,\mathbb{Z})$-action on n-torus is given. Our study generalizes the description of 2-fold self-joining of GL$(2,\mathbb{Z})$ action on $\mathbb{T}^2$ due to K. Park. We show that any joining is a linear combination of the Haar measures on subgroups of special form.

The purpose of this paper is to study self-joinings of n-torus automorphism group Aut$(\mathbb{T}^n)$.

We consider the group $G = \mathrm{GL}(n,\mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^n)$ of invertible matrices $t = (t^j_k)^n_{j,k=1}$ with integral coefficients and $\det t = \pm 1$. The group G acts on the n-torus $\mathbb{T}^n = \{\xi = (\xi^1,\dots,\xi^n)\}$ by the transformations

$$\Psi^t : \mathbb{T}^n \to \mathbb{T}^n, \qquad (\Psi^t\xi)^j := \sum_{k=1}^{n} t^j_k \xi^k \pmod 1.$$

Evidently, Ψ^t preserves the normalized Haar measure on $\mathbb{T}^n$.

We can define the *diagonal action*

$$\Psi_{(m)} := \{\Psi^t_{(m)} := \underbrace{\Psi^t \times \ldots \times \Psi^t}_{m \text{ multipliers}} : t \in \mathrm{GL}(n,\mathbb{Z})\}$$

on $(\mathbb{T}^n)^{\times m} := \prod_{i=1}^{m} \mathbb{T}^n$. Any element $x \in (\mathbb{T}^n)^{\times m}$ is a matrix (ξ^j_i), where i is the number of a multiplier in the direct product, $1 \le i \le m$, and ξ^j_i is the jth coordinate of the vector $x_i = (\xi^1_i,\dots,\xi^n_i) \in \mathbb{T}^n$. Thus, we have

$$(\Psi^t_{(m)}\xi)^j_i = \sum_{k=1}^{n} t^j_k \xi^k_i.$$

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The phase space $(\mathbb{T}^n)^{\times m}$ of the action $\Psi_{(m)}$ can be represented in the form

$$(\mathbb{T}^n)^{\times m} = \mathbb{T}^{nm} = (\mathbb{T}^m)^{\times n} = \{x = (x^1, \dots, x^n) : x^j = (\xi_i^j)_{i=1}^m \in \mathbb{T}^m\}.$$

Thus, $\Psi_{(m)}^t$ can be thought of as a multiplication of $(x^1, \dots, x^n)$ by t:

$$\Psi_{(m)}^t x = \begin{pmatrix} t_1^1 & \dots & t_n^1 \\ \vdots & & \vdots \\ t_1^n & \dots & t_n^n \end{pmatrix} \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}.$$

Definition 1. We say that a Borel measure ν on $(\mathbb{T}^n)^{\times m}$ is m-fold *self-joining* of Ψ if the following conditions hold:

(1) ν is $\Psi_{(m)}$-invariant;
(2) the "one-dimensional" projections $\pi_i\nu(A_i) = \nu(\mathbb{T}^n \times \dots \times A_i \times \dots \times \mathbb{T}^n)$ of ν coincide with the Haar measure λ on $\mathbb{T}^n$.

Two-fold self-joinings of $\mathrm{GL}(2,\mathbb{Z})$-action on $\mathbb{T}^2$ was studied by K. Park in [2] (see also [1]). She has shown that any ergodic self-joining should satisfy a certain algebraic condition specified in the following theorem.

Theorem 2. *Any ergodic* 2*-fold self-joining of* $\mathrm{GL}(2,\mathbb{Z})$*-action* Ψ *on* $\mathbb{T}^2$*, which is not equal to* $\mu^{\times 2}$ *(*μ *is the Haar measure on* $\mathbb{T}^m$*), is supported on a manifold* $kx_1 - \ell x_2 = 0$.

We give another formulation of Park's theorem that is a bit closer to the context of this paper.

Theorem 3. *Let* ν *be an ergodic* 2*-fold self-joining of the action* Ψ, $\nu \neq \mu^{\times 2}$*. Then* $\mathrm{supp}(\nu) \subseteq H \times H$*, where* H *is some proper closed subgroup of* $\mathbb{T}^m$.

The main purpose of this paper is to generalize Park's theorem to the case of m-fold self-joinings, $m > 2$, and to give an explicit classification of ergodic self-joinings.

Remark 4. Note that these results have some common features with the description of joinings of horocycle flows due to M. Ratner (see [3]). Namely, any non-trivial ergodic self-joining ν of a horocycle flow $\{u^t\}$ is supported on some manifold Ω such that the corresponding flow (Ω, ν, U^t) is transitive and is naturally isomorphic to some horocycle flow.

Further, it follows from Theorem 7 that all self-joining of $\mathrm{GL}(2,\mathbb{Z})$-action are the linear combinations of measures ν_H, where ν_H is Haar measure on $H \times H$, and the corresponding action of $\mathrm{GL}(2,\mathbb{Z})$ on $(H \times H, \nu_H)$ is again a subgroup in $\mathrm{Aut}(H \times H)$. Moreover, in both cases (horocycle flows and $\mathrm{GL}(2,\mathbb{Z})$-action) any ergodic self-joining is the measure uniformly distributed on the closure of an orbit, and, besides, there are countably many ergodic joinings.

Our first goal is to prove the following strengthening of Theorem 3.

Theorem 5. *If an ergodic m-fold self-joining ν of the action Ψ is not equal to $\mu^{\times n}$ then there exists a proper closed subgroup H of $\mathbb{T}^m$ such that*

$$\operatorname{supp}(\nu) \subseteq H^{\times n} = \underbrace{H \times \ldots \times H}_{n \text{ multipliers}}.$$

Proof. To simplify computations we suppose that $n = 3$. Let us consider two special classes of automorphisms of $\mathbb{T}^3$. The first class is formed by the transformations $S_{i,j}$ given by the equation $S_{i,j}x^j = x^j + x^i$, $S_{i,j}x^k = x^k$ as $k \neq j$. The second one consists of the transformations $F_{i,j}$ which permute ith and jth coordinates. For instance, transformations $S_{1,2}$ and $F_{1,2}$ have the following form:

$$S_{1,2} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let ν be a Ψ-invariant measure on $(\mathbb{T}^m)^{\times 3}$. Suppose that $\nu \neq \mu^{\times 3}$, where μ is the Haar measure on $\mathbb{T}^m$. We will show that $\operatorname{supp}(\nu) \subseteq H^{\times 3}$ for some proper closed subgroup H of $\mathbb{T}^m$. Let us represent the phase space $(\mathbb{T}^m)^{\times 3}$ as the product

$$(\mathbb{T}^m)^{\times 3} = Y \times \mathbb{T}^m, \quad \text{where } Y = \{(x^1, x^2)\} = (\mathbb{T}^m)^{\times 2},$$

and ν as the integral of fiber measures: $\nu(\cdot) = \int_Y \nu_y(\cdot)\, d\gamma(y)$, where ν_y may be considered as probability measures on $\mathbb{T}^m$.

We know that ν is invariant under the action of $S_{1,3}$ and $S_{2,3}$. Thus, since the partition into the sets $\mathbb{T}^m \times \{y\}$ is also $S_{i,3}$-invariant, the measure ν_y must be γ-almost surely invariant under the translations by x^2 and x^3 (see the definition of $S_{i,j}$). Recall now that ν_y is a Borel measure, thus, the foregoing discussion implies that ν_y is invariant under the translation by any element of the closed subgroup $V(y)$ of $\mathbb{T}^m$ generated by x^2 and x^3.

Suppose that $\nu \neq \mu^{\times 3}$. Then there exists a set $C \subseteq Y$ of positive γ-measure such that $\forall y \in C$ the measure $\nu_y \neq \mu$. Since there are only countably many closed subgroups of torus, the set C can be chosen so that for $y \in C$ the following conditions hold:

$$V(y) \equiv V \subset \mathbb{T}^m \text{ and } V \neq \mathbb{T}^m.$$

Note that $C \subseteq V \times V$. Let $\mu(\cdot|V)$ denote the Haar measure on the subgroup V. Since the action of V by translations on itself is unique ergodic, the measure ν_y must have the form

$$\nu_y(\cdot) = \int\limits_{\mathbb{T}^m/V(y)} \mu(\cdot \mid x^3 + V(y))\, dc_y(\bar{x}^3), \tag{1}$$

where $\mu(\cdot \mid x^3 + V(y))$ is the translation of $\mu(\cdot|V(y))$: $\mu(A \mid x^3 + V(y)) := \mu(A - x^3 \mid V(y))$. Now we make use of the fact that ν is invariant with respect to $F_{2,3}$. Fixing a set $A \subseteq \mathbb{T}^m$ we have $\nu(V \times V \times A) = \nu(V \times A \times V)$. If $\operatorname{supp}\nu \subset V \times V \times V$, then the theorem is proved. Thus, we assume that $A \cap V = \emptyset$.

Let us consider a point $(v, a) \in V \times a$ and denote by W the minimal closed subgroup of $\mathbb{T}^m$ that includes V and a, i. e. $W = V(v, a)$. Then from (1) we obtain:

$$\nu(V \times A \times V) = \int\limits_{V \times A} d\gamma(v, a) \int\limits_{\mathbb{T}^m/V} \mu(V \mid x^3 + W)\, dc_{(a,v)}(\bar{x}^3).$$

If $\#W/(V \cap W) = \infty$ then $\mu(V \mid x^3 + W) = 0$. Thus, for γ-a. a. (a, v) the point a belongs to a certain finite extension of the subgroup $V \cap W$. Taking into account countability of all finite extensions we conclude that for some finite extension H the value $\nu(V \times H \times V) > 0$. (We can extend H so that H becomes a finite extension of V.) Moreover, $\nu(H \times H \times H) > 0$. Recall that the measure ν is supposed to be ergodic. Thus, since $H \times H \times H$ is Ψ-invariant, $\operatorname{supp}(\nu) \subseteq H \times H \times H$ and the proof is completed. □

Let us give one of the possible generalizations of the previous result.

Theorem 6. *Let G be a compact Abelian group which is not a finite extension of some proper subgroup and let Ψ be the natural action of the group* $\mathrm{GL}(n, \mathbb{Z})$ *on $G^{\times n}$. Then any ergodic Ψ-invariant measure ν on $G^{\times n}$ possesses the following property*:

$$\operatorname{supp}(\nu) \subseteq H \times \ldots \times H \quad (n \ \textit{multipliers}),$$

where H is a proper closed subgroup of G. Since $H \times \ldots \times H$ is invariant under the diagonal action of automorphism group $\mathrm{Aut}(G)$ *of G, the same is true for* $\mathrm{Aut}(G)$.

Theorem 5 gives no description of the ergodic joinings of $\mathrm{GL}(n, \mathbb{Z})$. Let us give a simple example of joining (see Fig. 1) that is not equal to the Haar measure on some $H^{\times n}$. Namely, let us consider the measure ν_3 on $\mathbb{T}^2 \times \mathbb{T}^2$ which is uniformly distributed on the manifold $(\Delta_2 \times \Delta_2) \setminus \Delta$, where $\Delta := \{(\xi, \xi) : \xi \in \mathbb{T}\}$ is the *diagonal* and $\Delta_2 := \Delta \sqcup (\Delta + (\frac{1}{2}, 0))$. Evidently, ν_3 is an ergodic joining and ν_3 does not coincide with the Haar measure on $\Delta_2 \times \Delta_2$.

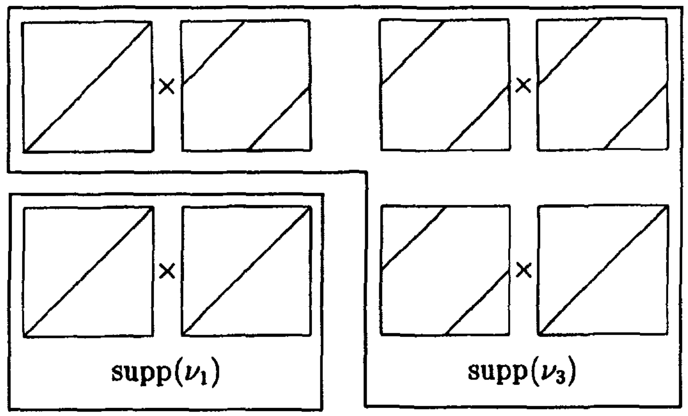

Fig. 1

Now we turn to the classification of ergodic $\mathrm{GL}(n,\mathbb{Z})$-invariant measures and, as a particular case, ergodic self-joinings.

Let us consider a proper closed subgroup H of the torus $\mathbb{T}^m$ and its connected component of zero U. Setting $F := H/U$, we use the designation $\bar{f}$ for the coset of $f \in H$ with respect to U. Let $\langle \bar{f}^1, \dots, \bar{f}^n \rangle$ denote the subgroup of F generated by elements $\bar{f}^1, \dots, \bar{f}^n$.

The main result of this paper is the following

Theorem 7. *Any ergodic* $\mathrm{GL}(n,\mathbb{Z})$*-invariant Borel measure* ν *on* $(\mathbb{T}^m)^{\times n}$ *is the uniform distribution on a manifold of the following form:*

$$\mathcal{O}_H + U^{\times n}, \quad \mathcal{O}_H = \left\{ (\bar{f}^1, \dots, \bar{f}^n) \in F^{\times n} : \langle \bar{f}^1, \dots, \bar{f}^n \rangle = F \right\}, \qquad (2)$$

where H is a proper closed subgroup of $\mathbb{T}^m$ such that $F = H/U$ has n generators.

In particular, ν is ergodic m-fold self-joining of $\mathrm{GL}(n,\mathbb{Z})$*-action Ψ on $\mathbb{T}^n$ iff ν is the uniform distribution on a manifold of the form* (2) *such that all one-dimensional projections of U coincide with $\mathbb{T}^1$.*

Let $\nu \neq \mu^{\times n}$ be a joining. By virtue of Theorem 5 we have $\mathrm{supp}(\nu) \subseteq H^{\times n}$, where $H \subseteq \mathbb{T}^m$ and the dimension of H is less than m. The naïve idea of the proof is try to restrict Ψ to $H^{\times n}$ and use the induction on the dimension m. However, two serious problems arise. The first is that H may not be connected (more exactly, H can be a finite extension of some proper subgroup), and, therefore, the direct application of Theorem 5 to $\Psi|_H$ is impossible.

Let U be the connected component of zero of the subgroup H, $U \cong \mathbb{T}^{m_1}$, $m_1 < m$. Since U is connected, the action Ψ restricted to $H^{\times n}$ is a skew product over $H^{\times n}/U^{\times n}$ (and $H^{\times n}/U^{\times n}$ is a factor of Ψ). Namely, we can correctly define the action $\bar{\Psi}$ on $H^{\times n}/U^{\times n}$ by the equation

$$\bar{\Psi}(f + U^{\times n}) = \Psi f + U^{\times n}, \quad f \in H^{\times n}.$$

$\bar{\Psi}$ preserves the probability measure $\bar{\nu}$ on $H^{\times n}/U^{\times n}$ which is defined in the usual way: $\bar{\nu}(\bar{f}) = \nu(f+U^{\times n})$, where $\bar{f}$ is the element of the quotient group $H^{\times n}/U^{\times n}$ corresponding to the point $f \in H^{\times n}$. Clearly, ergodicity of ν implies ergodicity of $\bar{\nu}$. Since the space $H^{\times n}/U^{\times n}$ is finite, ergodicity of $\bar{\nu}$ means, simply, that $\bar{\nu}$ is uniformly distributed on some orbit of $\bar{\Psi}$. Thus, calculating orbits of $\bar{\Psi}$, we obtain the description of $\bar{\Psi}$-invariant ergodic measures.

Let us restrict the action Ψ on some orbit $\mathcal{O}$ (which is considered as a submanifold of $(\mathbb{T}^m)^{\times n}$) and fix a point $\bar{f} \in \mathcal{O}$. The ergodic joining ν with non-trivial component in $\mathcal{O}$ can be described by its restriction to $f + U^{\times n}$. Therefore we need to characterize all the measures on $f + U^{\times n}$ that is invariant under the stabilizer $\tilde{\Psi}_1$ of $\bar{f}$ in Ψ.

The second obstacle of applying Theorem 5 is that the action $\tilde{\Psi}_1$ is not an action by automorphisms. Nevertheless, choosing a point in $f + U^{\times n}$ that will be considered as "zero" we can assume that $\tilde{\Psi}_1$ includes a subgroup generated by the automorphisms $S_{i,j}(k)$ for some $k \in \mathbb{N}$.

Note that we cannot require from $\tilde{\Psi}_q$ more than we have. Therefore, we must use as the inductive step a statement that is stronger than Theorem 5.

Lemma 8. *Let Ψ be an action of some subgroup of* $\mathrm{GL}(n,\mathbb{Z})$ *on $(\mathbb{T}^m)^{\times n}$. Suppose that there exists $k \neq 0$ such that*

$$\forall\, i \neq j \quad S_{i,j}(k) \in \Psi,$$

where $S_{i,j}(k)$ is given by the equations

$$S_{i,j}(k)x^j := kx^i + x^j, \quad S_{i,j}(k)x^l := x^l \quad for \quad l \neq j.$$

Then any ergodic Ψ-invariant Borel measure ν that is not equal to $\mu^{\times n}$ is lokated on some proper closed subgroup $H^{\times n}$:

$$\mathrm{supp}(\nu) \subseteq H^{\times n} = \underbrace{H \times \ldots \times H}_{n \quad multipliers}.$$

Proof. For the beginning, let us consider the case of $\mathrm{GL}(2,\mathbb{Z})$-action. We say that the point $x \in H$ (where H is a closed subgroup) is *irrational* if x does not belong to some proper closed subgroup of H. Let $\mathcal{I}(H)$ be the set of all irrational elements of H,

$$\mathcal{I}(H) := H \setminus \bigcup_{W<H} W.$$

The partition of $\mathbb{T}^m$ into the sets $\mathcal{I}(H)$, $H \le \mathbb{T}^m$, is a countable partition of $\mathbb{T}^m$ into Borel sets. Let us denote by $\mathcal{R}(H) := H \setminus \mathcal{I}(H)$ the set of all *rational* elements of H.

Let $\nu \ne \mu^{\times n}$ be an ergodic Ψ-invariant measure. Then the measures ν and $\mu^{\times n}$ must be mutually singular. As in the proof of Theorem 5, we represent ν as the integral of fiber measures:

$$\nu = \int_{\mathbb{T}^m} \nu_{x^1}\, d\gamma(x^1).$$

Suppose that $x^1 \in \mathcal{I}(\mathbb{T}^m)$. It is clear that $V(kx^1) = \mathbb{T}^m$ (see the notation from the proof of Theorem 5). Thus, $\nu_{x^1} = \mu$. If $\gamma(\mathcal{I}(\mathbb{T}^m)) > 0$, then repeating the same argument for the second coordinate we have $\gamma(\cdot|\mathcal{I}(\mathbb{T}^m)) = \mu$. Hence, the measures ν and $\mu \times \mu$ cannot be mutually singular and we come to the contradiction. Thus,

$$\operatorname{supp}(\nu) \subseteq \mathcal{R}(\mathbb{T}^m) \times \mathcal{R}(\mathbb{T}^m).$$

Evidently, the set $\mathcal{R}(\mathbb{T}^m) \times \mathcal{R}(\mathbb{T}^m)$ is the union of all sets $\mathcal{I}(G)$, where G is a subgroup of $(\mathbb{T}^m)^{\times 2}$ with the projections $H_i < \mathbb{T}^m$ (such subgroups G will be called *admissible*).

Since any transformation $\psi \in \Psi$ is an automorphism, $\psi(\mathcal{I}(G)) = \mathcal{I}(\psi(G))$. Thus, the partition $\{\mathcal{I}(G) : G \le \mathbb{T}^m \times \mathbb{T}^m\}$ is Ψ-invariant and we can consider the factor-action corresponding to this measurable partition which we denote by the same symbol Ψ.

Our goal is to show that for any admissible subgroup G with the transversal projections ($\langle H_1, H_2\rangle = \mathbb{T}^m$) the orbit of $\mathcal{I}(G)$ is infinite with respect to the factor-action. Then $\nu(\mathcal{I}(G)) = 0$ and, hence, there exists a subgroup G with the properties $\nu(\mathcal{I}(G)) > 0$ and $H := \langle H_1, H_2\rangle < \mathbb{T}^m$, where $H_i = \pi_i G$. Thus, $\nu(H \times H) > 0$ and the proof is completed.

So, let us fix an admissible subgroup $G < \mathbb{T}^m \times \mathbb{T}^m$ with the transversal projections and suppose at first that G is connected. It is enough for our purposes to consider the so-called *local group* G_{loc} instead of G. In other words, we have only to consider the behavior of G near the zero. Indeed, if the local group G_{loc} is not fixed by some transformation, then G is not fixed as well. Let us consider some coordinates $\eta^1, \dots, \eta^r$ on G choosing matrices $A = (a_i^j)$ and $B = (b_i^j)$, $1 \le i \le r$, $1 \le j \le m$, such that

$$(x, y) \in G_{\mathrm{loc}} \quad \Longleftrightarrow \quad x^j = \sum_{k=1}^{r} a_k^j \eta^k, \quad y^j = \sum_{k=1}^{r} b_k^j \eta^k, \quad \eta \in \mathbb{T}^r.$$

We will use the following notation: $G_{\mathrm{loc}} = G_{\mathrm{loc}}(A, B)$.

Let us consider possible types of transformations of the matrix (A, B). The transformation $(A, B) \mapsto U(A, B)$, where U is invertible, corresponds to a change of coordinates in $\mathbb{T}^r$. Thus, $G_{\mathrm{loc}}(U(A, B)) = G_{\mathrm{loc}}(A, B)$. Further, multiplying (A, B) on the right by the matrix $\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$, $V \in \mathrm{GL}(m, \mathbb{Z})$, we make some coordinate change in $\mathbb{T}^m$. Finally, the action of $t \in \Psi$ on G_{loc} can be represented in the form

$$\Psi^t : G(A, B) \mapsto G\left((A, B)\begin{pmatrix} t_1^1\mathrm{Id} & t_1^2\mathrm{Id} \\ t_1^2\mathrm{Id} & t_2^2\mathrm{Id} \end{pmatrix}\right),$$

where $\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that, since H_1 and H_2 are transversal, both A and B is not equal to 0. Let G' be the image of the subgroup G under the transformation $S_{1,2}(k)$ corresponding to $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. The matrix describing G'_{loc} has the form $(A, kA + B)$. Let us prove that there is no invertible U with the property $A = UA$, $kA + B = UB$. Suppose that such a matrix exists. Since $H_2 < \mathbb{T}^m$, there exists a vector $a \in \mathbb{R}^m$ such that $Ba = 0$. Then $Aa = \frac{1}{k}(U - 1)Ba = 0$. So, if we consider the matrices A and B as operators in Euclidian space, we can conclude that they kernels intersect, hence, $\langle H_1, H_2\rangle < \mathbb{T}^m$. Recall now that H_1 and H_2 are transversal, whence, $\langle H_1, H_2\rangle = \mathbb{T}^m$. This contradiction proves our hypothesis. Thus, $G'_{\mathrm{loc}} \neq G_{\mathrm{loc}}$ and $S_{1,2}(qk)G_{\mathrm{loc}} \neq G_{\mathrm{loc}}$ for any $q \in \mathbb{Z}$. Since the property of transversality is conserved by the transformation $S_{1,2}(k)$, all the subgroups $S_{1,2}(qk)G$ are distinct, whence, the orbit of $\mathcal{I}(G)$ is infinite according to the factor-action.

If G is not connected we have to consider the connected component of zero of G_{loc} and apply the preceding. Since all the local subgroups $S_{1,2}(qk)G_{\mathrm{loc}}$ are distinct, the orbit of $\mathcal{I}(G)$ is also infinite.

So, Lemma 8 is proved for the action of $\mathrm{GL}(2, \mathbb{Z})$. If $n > 2$ then a local subgroups G_{loc} can be parametrized by a matrix $\mathcal{A} = (A_1, A_2, \ldots, A_{n-1}, B)$. A transformation $S_{j,n}(k)$ changes the nth block in $\mathcal{A}$ to the matrix $kA_j + B$. If we suppose that for fixed k all the matrices $kA_j + B$ can be reduced to B, then we find a common eigenvector of the collection $\{A_1, \ldots, A_{n-1}, B\}$ corresponding to zero. But this contradicts to the condition

$$\langle \mathbb{T}^r A_1, \ldots, \mathbb{T}^r A_{n-1}, \mathbb{T}^r B\rangle = \mathbb{T}^m.$$

Thus, the condition of transversality of the projections H_i of a group G implies that $\nu(\mathcal{I}(G)) = 0$. Whence, since ν is non-trivial, for some proper closed subgroup $H < \mathbb{T}^m$ we have $\mathrm{supp}(\nu) \subseteq H^{\times n}$. $\square$

Let U be a connected closed subgroup of $\mathbb{T}^m$, H be some finite extension of U and $F := H/U$. Recall that we consider the factor-action $\bar{\Psi}$ which is defined as the restriction of Ψ on $H^{\times n}$ to the invariant partition of $H^{\times n}$ into cosets $f + U^{\times n}$, where $f \in F^{\times n}$.

Definition 9. We call a manifold $\mathcal{O} + U^{\times n}$ a *U-orbit* of the action $\Psi \subseteq \mathrm{GL}(n,\mathbb{Z})$ if $\mathcal{O}$ is an orbit of the factor-action $\bar{\Psi}$.

Proposition 10. *Suppose that $\Psi \subseteq \mathrm{GL}(n,\mathbb{Z})$ acts on $(\mathbb{T}^m)^{\times n}$ and contains all the transformations $S_{i,j}(k)$, $1 \le i,j \le n$, where k is some fixed integer. Then ν is ergodic Ψ-invariant Borel measure on $(\mathbb{T}^m)^{\times n}$ iff ν is the uniform distribution on a U-orbit of Ψ.*

Proof. The proof is by induction on the dimension of torus $\mathbb{T}^m$. The case $m = 1$ is trivial. Let $m > 1$.

If $\nu = \mu^{\times n}$, then there is nothing to prove.

Suppose that $\nu \ne \mu^{\times n}$. Then by virtue of Lemma 8 there exists a proper closed subgroup $W < \mathbb{T}^m$ such that $\mathrm{supp}(\nu) \subseteq W^{\times n}$. If W is connected, then we can use the induction hypothesis. Thus, the restriction of ν on $W^{\times n}$ possesses the following property: $\mathrm{supp}(\nu)$ is a U-orbit of the action $\Psi\big|_{W^{\times n}}$. Then $\mathrm{supp}(\nu)$ is a U-orbit of Ψ.

Let W be not connected and $F := W/U$, where U is the connected component of zero in W. As is customary, let us consider the factor-action $\bar{\Psi}$ that is defined on $W^{\times n}/U^{\times n}$ by the equation

$$\bar{\Psi}(f + U^{\times n}) = \Psi(f) + U^{\times n}.$$

Denote by $\bar{\nu}$ the image of ν under the natural projection $\pi : W^{\times n} \to F^{\times n}$. Recall that $\bar{\nu}$ is ergodic and is the uniform distribution on some orbit $\mathcal{O}$ of the action $\bar{\Psi}$. For $\bar{f} \in \mathcal{O}$ consider the stabilizer $\tilde{\Psi}_1$ of the point $\bar{f}$ with respect to $\bar{\Psi}$.

Set $k := \#F$. Since $k\bar{f}^j = 0$ we can choose the points o^j in $f^j + U$ with the property $ko^j = 0$. Consider the new coordinates $u^j = x^j - o^j$ on $f + U^{\times n}$. Clearly, $(\bar{f} + U^{\times n}) \cong U^{\times n}$ and the transformations $S_{i,j}(k)$ become automorphisms.

Let us define the subgroup $\Psi_1 \subseteq \tilde{\Psi}_1$ generated by the transformations $S_{i,j}(k)$, $1 \le i,j \le n$ and consider the measure ν_1 that is the restriction of ν to the manifold $f + U^{\times n}$. It is obvious that ν is $\tilde{\Psi}_1$-invariant, hence, Ψ_1-invariant. The dynamical system $(\Psi_1, \bar{f}, \nu^1)$ satisfies the conditions of this proposition. Thus, by the induction hypothesis, the measure ν_1 is uniformly distributed on some V-orbit of the action Ψ_1.

From the definition of U-orbit and the foregoing discussion it follows that the measure ν is the uniform distribution on a manifold that is the union of translates of $V^{\times n}$. Since $\mathrm{supp}(\nu)$ is a subset of $W^{\times n}$, the $\mathrm{supp}(\nu)$ is again a V-orbit, since $\mathrm{supp}(\nu)$, considered as a subset of $W^{\times n}/V^{\times n}$, must be equal to an orbit of the factor-action $\bar{\Psi}$. $\square$

To complete the proof of Theorem 7, we have only to give the description of U-orbits of the $\mathrm{GL}(n,\mathbb{Z})$-action.

Proof of Theorem 7. Let us show that any U-orbit of Ψ has the form given by Eq. (2). Indeed, let H be the minimal subgroup of $\mathbb{T}^m$ with the property $\mathrm{supp}(\nu) \subseteq H^{\times n}$. Put $F := H/U$ and consider two vectors $\bar{f}, \bar{g} \in F^{\times n}$. We say that this vectors are *equivalent*, $\bar{f} \approx \bar{g}$, if there is a matrix $U \in \mathrm{GL}(n, \mathbb{Z})$ such that $\bar{g} = U\bar{f}$. In other words, $\bar{f} \approx \bar{g}$ iff f and g belong to the same orbit of the factor-action $\bar{\Psi}$ (see Definition 9).

Let us consider an orbit $\mathcal{O}$ of $\bar{\Psi}$ and $\bar{f} \in \mathcal{O}$. With the notation $\langle \bar{f} \rangle := \langle f^1, \dots, f^n \rangle$, let us show that for any other $\bar{g} \in \mathcal{O}$ the following condition holds: $\langle \bar{g} \rangle = \langle \bar{f} \rangle$. Indeed, since $\bar{g} = U\bar{f}$, we have $\forall j \;\; g^j \in \langle \bar{f} \rangle$. Thus, $\langle \bar{g} \rangle \subseteq \langle \bar{f} \rangle$, and analogously, $\langle \bar{f} \rangle \subseteq \langle \bar{g} \rangle$.

Conversely, suppose $\langle \bar{g} \rangle = \langle \bar{f} \rangle$. Without loss of generality we may assume that $\langle \bar{f} \rangle = F$. Using the finite Abelian groups' classification it is easy to show that

$$\bar{g} \approx \bar{f} \iff \langle \bar{g} \rangle = \langle \bar{f} \rangle.$$

Then $\bar{g} \in \mathcal{O}$. Thus, the orbit $\mathcal{O}$ consists of the elements $\bar{g}$ with the property $\langle \bar{g} \rangle = F$ and F must have n generators (in our case, $f^1, \dots, f^n$). $\square$

Using the same technique one can prove the following theorem (see [4]).

Theorem 11. *Let Ψ be the action of the group of finite invertible matrix on the direct power $F^{\times I} = \prod_{i \in I} F$ of a finite Abelian group F. Suppose that I is infinite. Then the ergodic Ψ-invariant measures on F^I are the Haar measures on the subgroups*

$$H^{\times I} = \prod_{i \in I} H, \quad H \leq F.$$

In particular, a Borel measure ν on $(F^{\times m})^{\times I}$ is an ergodic m-fold self-joining of the automorphism group of $F^{\times I}$ iff ν is the Haar measure on $L^{\times I}$, where $L \leq F^{\times m}$ is a subgroup with the standard one-dimensional projections.

For example, the automorphism group of $\mathbb{Z}_2^{\times I}$ has two ergodic two-fold self-joinings: $\lambda \times \lambda$ and Δ, where $\Delta(A \times B) := \lambda(A \cap B)$ and λ is the Haar measure on $\mathbb{Z}_2^{\times I}$. The automorphism group of $\mathbb{Z}_p^{\times I}$ (p is prime integer) has p self-joinings: $\lambda \times \lambda$ and the measures

$$\{(\mathrm{Id} \times S_a)\Delta : a \in \mathbb{Z}_p \setminus \{0\}\},$$

where $S_a x^j = a x^j$. Further examples could be found in [4].

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