On spectral properties of iceberg transformations

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Spectral invariants

Let T be an invertible measure preserving transformation of the standard Lebesgue space $(X, \mathcal{A}, \mu)$, $X = [0, 1]$.
The Koopman operator

$$\widehat{T}\colon\ L^2(X, \mu) \to L^2(X, \mu)\colon\ f(x) \mapsto f(Tx)$$

Spectral invariants of T are the

- maximal spectral type σ on $S^1 = \{z \in \mathbb{C}\colon\ |z| = 1\}$ and the
- multiplicity function $\mathcal{M}(z)\colon\ S^1 \to \mathbb{N} \sqcup \{\infty\}$.

Usually we study $\widehat{T}$ on the space of functions with zero mean.

Rank one transformations

Definition. T is called a *rank one* transformation if there exist a sequence of partitions

$$\xi_n = \{B_n, TB_n, T^2B_n, \ldots, T^{h_n-1}B_n, E_n\},$$

identified with Rokhlin towers, such that $\mu(\bigcup_{j=0}^{h_n-1} T^jB_n) \to 1$ and for any measurabe set A there exist ξ_n-measurable sets A_n with $\mu(A \triangle A_n) \to 0$.

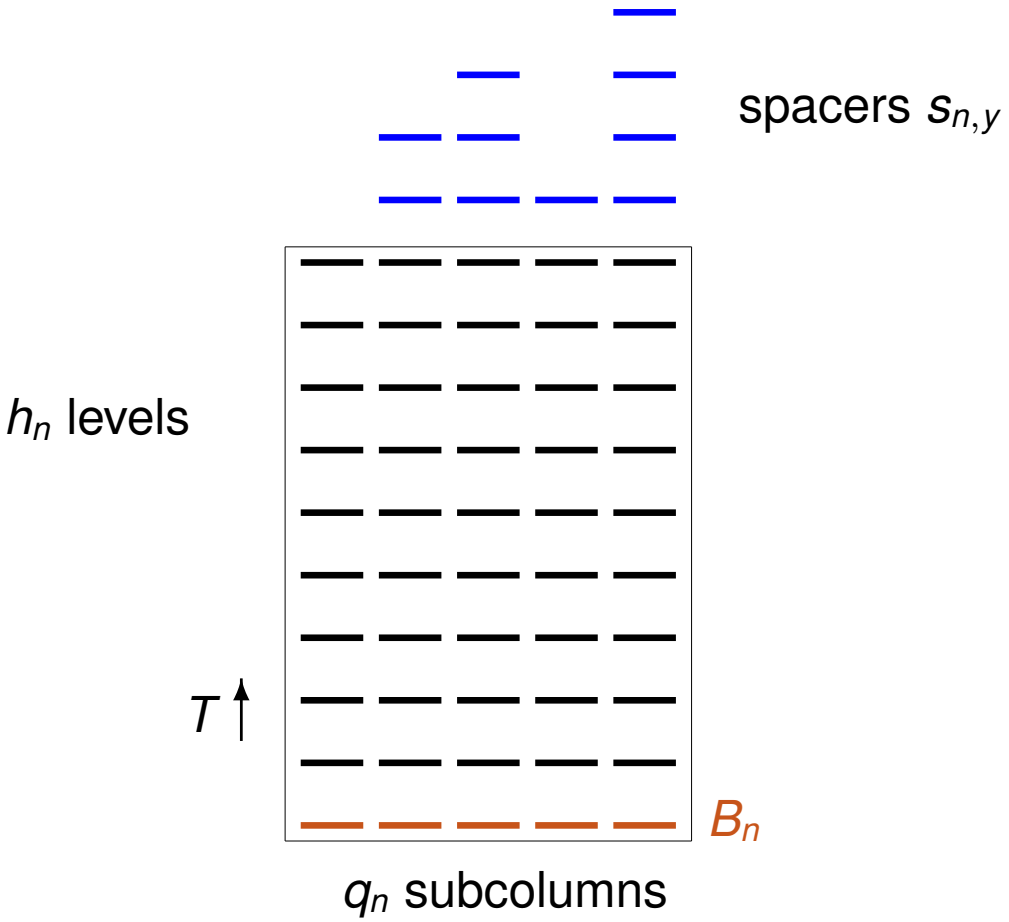
Rank one transformations: Symbolic definition

Any rank one transformation can be consructed in the following way. Starting from a word W_{n_0} consider the sequence of words W_n given by

$$W_{n+1} = W_n 1^{s_{n,1}} W_n 1^{s_{n,2}} W_n \dots 1^{s_{n,q_n}} W_n,$$

where symbol "1" is used to create *spacers* between words, and parameters $s_{n,1}$ are fixed in advance.

Cutting-and-stacking construction



$$\prod_{n=1}^{\infty} \frac{h_{n+1}}{q_n h_n} < \infty$$

Rank one systems: Simplicity of spectrum

Lemma (*Katok, Stepin*). Let U be a unitary operator on a separable Hilbert space H, and let σ and $M(z)$ be the spectral type and the multiplicity function of U.
If $M(x) \geq m$ on a set of positive σ-measure then there exist m orthogonal vectors $f_1, \ldots, f_m$ such that for any cyclic space $Z \subseteq H$ and any $g_1, \ldots, g_m \in Z$, $\|g_i\| \equiv a$, the following is true

$$\sum_{i=1}^{m} \|f_i - g_i\|^2 \geq m(1 + a^2 - 2a/\sqrt{m})$$

Spectral invariants

A transformation T has simple spectrum, i.e. $M(z) \equiv 1$, iff there exists an element $f \in L^2(X, \mu)$ (*cyclic vector*) such that

$$L^2(X, \mu) = \overline{\mathrm{Span}}(\{\widehat{T}^k f \colon\ k \in \mathbb{Z}\}).$$

In this case $\sigma_f \sim \sigma$, where σ_f is defined by

$$\int_{S^1} z^k \, d\sigma_f = \left\langle T^k f,\ f \right\rangle.$$

Theorem. Both rank one transformations and rank one flows are ergodic and have simple spectrum.

Rank one systems: Spectral type

Question. Is the following true: Any rank one transformation has the spectral type which is *singular* with respect to the Lebesgue measure λ on S^1?

$$\sigma_f \perp \lambda$$

Question (Banach). Does there exist an automorphism with spectral multiplicity 1 and Lebesgue spectral type?

Generalized Riesz products

Let us define polynomials

$$P_n(z) = \frac{1}{\sqrt{q_n}} \sum_{y=0}^{q_n-1} z^{\omega_n(y)},$$

where $\omega_n(y)$ are return times to the base B_n of n-th tower,

$$\omega_n(y) = yh_n + s_{n,0} + \ldots + s_{n,y-1}$$

for an orbit starting at the base B_{n+1} of the first subcolumn.

Generalized Riesz products

The spectral measure σ_f of a function $f \in L^2(X, \mu)$ constant on the levels of a tower with index n_0 is given (up to a constant multiplier) by the infinite product

$$\sigma_f = |\widehat{f}_{(n_0)}|^2 \prod_{n=n_0}^{\infty} |P_n(z)|^2,$$

converging in the weak topology.

Applications to rank one systems

Theorem (Bourgain, 1993). Ornstein rank one transformations have singular spectral type.

Theorem (Klemes, 1994). A class of staircase constructions is of singular spectral type.

Polynomials with Littlewood type coefficient constraints

$$\mathcal{K}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k \colon \ |a_k| \equiv 1 \right\}.$$

$$\mathcal{L}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k \colon \ a_k \in \{-1,\ 1\} \right\}.$$

$$\mathcal{M}_n = \left\{ P(z) = \tfrac{1}{\sqrt{n}} (z^{\omega_1} + z^{\omega_2} + \ldots + z^{\omega_n}) \colon \ \omega_j \in \mathbb{Z},\ \omega_j < \omega_{j+1} \right\}.$$

Flatness phenomenon

Question (Littlewood, 1966). Is the following true?
For any $\varepsilon > 0$ there exists an *ultra-flat* polynomial with unimodular coefficients $P(z) \in \mathcal{K}_n$ such that

$$\forall z \in S^1 \qquad \big||P(z)| - 1\big| < \varepsilon.$$

Theorem (Kahane, 1980). The answer is "yes" with the speed of convergence

$$\varepsilon_n = O(n^{-1/17}\sqrt{\ln n}).$$

Flatness phenomenon

Question (*open*).
Is it possible to find an ultra-flat polynomial in $\mathcal{L}_n$?

Question (*open*).
Is it possible to find an L^p-flat polynomial in $\mathcal{M}_n$?

Notice that polynomials $P_n(z)$ in the generalized Riesz product generated by rank one transformations are in class $\mathcal{M}_{q_n}$.

Flat exponential sums with coefficients in $\{0, 1\}$

$$\mathcal{M}_q^{\mathbb{R}} = \left\{ \mathcal{P}(t) = \tfrac{1}{\sqrt{q}} \sum_{y=0}^{q-1} \exp(2\pi i\, t\omega(y) \colon\ \omega(y) \in \mathbb{R} \right\}.$$

Theorem. The answer is "yes" in class $\mathcal{M}_q^{\mathbb{R}}$. For any $0 < a < b$ and $\varepsilon > 0$ there exists a sum $\mathcal{P}(t) \in \mathcal{M}_q$ which is ε-flat in $L^1(a, b)$ (and $L^2(a, b)$),

$$\left\| \left| \mathcal{P}(t) \right|_{(a,b)} | - 1 \right\|_1 < \varepsilon.$$

Flat exponential sums with coefficients in $\{0, 1\}$

And

$$\mathcal{P}(t) = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} \exp(2\pi i\, t\omega(y),$$

$$\omega(y) = m\frac{q}{\beta^2} e^{\beta y/q},$$

where $m > 0$, $\beta^{-1} \in \mathbb{N}$ and q ranges over a set $\mathcal{Q}_{\varepsilon,a,b}(\beta, \varepsilon, m)$ of positive density in $\mathbb{Z}$.

Finite spectral multiplicity

Question. Does there exist an automorphism with *finite spectral multiplicity* and *absolutely continuous spectral type*?

Theorem (Guenais, 1998). Connection between Littlewood-type problem in $\mathcal{L}_n$ (coefficients ± 1) and the spectral properties of Morse cocycles.

Theorem (Downarowicz, Lacroix, 1998). If all continuous binary Morse systems have singular spectra then the merit factors of binary words are bounded (the Turyn's conjecture holds).

Rotation operator and IET on finite words

Let us define *rotation operator* ρ_α on finite words:

$$\rho_\alpha(W_{(1)}W_{(2)}) = W_{(2)}W_{(1)} \quad \text{if} \quad |W_{(1)}| = \alpha$$

Actually, ρ_α is an interval exchange transformation (IET).

Example: $\rho_1(\mathrm{CAT}) = \mathrm{ATC}, \quad \rho_2(\mathrm{CAT}) = \mathrm{TCA}.$

Iceberg map: Symbolic definition

Rotated words concatenation procedure:

$$W_{n+1} = \rho_{\alpha_{n,0}}(W_n)\rho_{\alpha_{n,1}}(W_n)\dots\rho_{\alpha_{n,q_n-1}}(W_n).$$

For example, if W_1 is the word "CAT", $q_1 = 6$ and $(\alpha_{1,0}, \alpha_{1,1}, \dots, \alpha_{1,q_1-1}) = (0, 1, 2, 2, 0, 1)$ then

$$\mathrm{CAT} \mapsto \mathrm{CAT.ATC.TCA.TCA.CAT.ATC} = W_2$$

Iceberg map: Symbolic definition

At the next step we rotate the word W_2.
The following table shows positions of cutting ($\times$)

$$\mathrm{CATATCT}_{\times}\mathrm{CATCACATATC}$$
$$\mathrm{CATA}_{\times}\mathrm{TCTCATCACATATC}$$
$$\mathrm{CATATCTCATC}_{\times}\mathrm{ACATATC}$$

used to create the word

$$W_3 = \mathrm{CATCACATATC}\,|\,\mathrm{CATATCT}\ .\ \mathrm{TCTCATCACATATC}\,|\,\mathrm{CATA}\ .\ \mathrm{ACATATC}\,|\,\mathrm{CATATCTCATC}$$

Iceberg map: Cutting-and-stacking

Suppose now that symbols $\{\mathrm{C}, \mathrm{A}, \mathrm{T}\}$ correspond to a partition

$$X = P_{\mathrm{C}} \sqcup P_{\mathrm{A}} \sqcup P_{\mathrm{T}}$$

and

$$\mu(P_{\mathrm{C}}) = \mu(P_{\mathrm{A}}) = \mu(P_{\mathrm{T}}) = \frac{1}{3}.$$

Definition. Let us draw all the rotations of the word CAT in a way used to draw a Rokhlin tower, placing same letters to the same level. This picture is called *iceberg*.
The level "C" is called the *base level* of the iceberg.

Iceberg

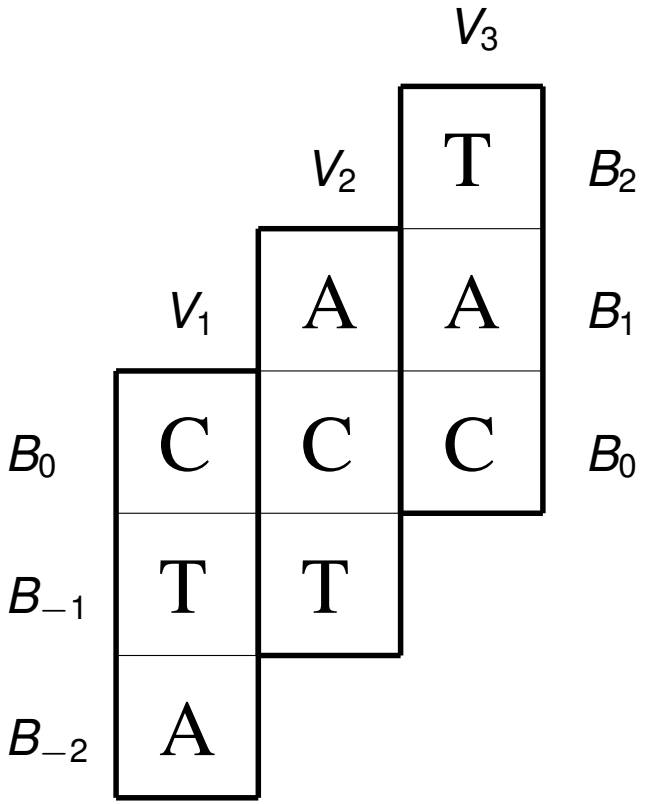


Figure: Iceberg associated with the word CAT

Iceberg

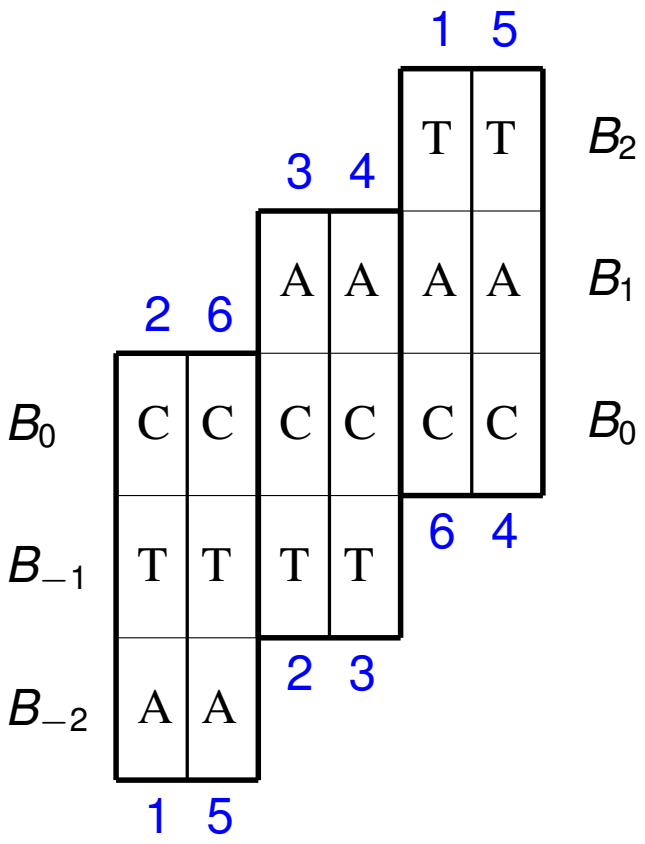


Figure: Poincaré map for the iceberg corresponding to sequence CAT.ATC.TCA.TCA.CAT.ATC

Iceberg

The basic idea of iceberg is to guess that a m.p.t. T maps each elementary set (shown as a square) to the upper set with small deviation. For example, if we split the column V_3 into letter-marked sets, $V = V_{3,\mathrm{C}} \sqcup V_{3,\mathrm{A}} \sqcup V_{3,\mathrm{T}}$, then we require:

$$\mu(TV_{3,\mathrm{C}} \mid V_{3,\mathrm{A}}) \approx 1 \quad \text{and} \quad \mu(TV_{3,\mathrm{A}} \mid V_{3,\mathrm{T}}) \approx 1.$$

Remark that if levels of the iceberg are $B_{-2}, B_{-1}, B_0, B_1, B_2$ then $P_{\mathrm{C}} = B_0$, $P_{\mathrm{A}} = B_1 \cup B_{-2}$, $P_{\mathrm{T}} = B_2 \cup B_{-1}$.

Spectral properties of random iceberg transformation

Theorem. Let T be an iceberg transformation given by uniform i.i.d. random rotations $\alpha_{n,k}$, and suppose that $q_n \gg h_n$ grows sufficiently fast. The following properties hold a.s.

(i) T is of $1/4$-local rank,

(ii) T has simple spectrum,

(iii) $\sigma * \sigma \ll \lambda$, where σ is the spectral type of $\hat{T}$ and λ is Lebesgue measure on S^1,

(iv) For a dense set of functions f with zero mean $\forall \varepsilon > 0$

$$\langle T^t f, f \rangle = O(t^{-1/2+\varepsilon}).$$

Icicle

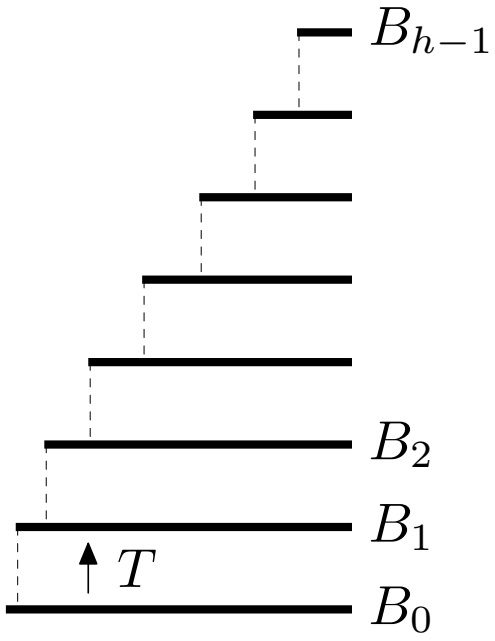


Figure: An icicle is a sequence of disjoint sets $\{B_0, B_1, \ldots, B_{h-1}\}$ such that $B_{j+1} \subseteq TB_j$ and $B_j \in \mathcal{A}$.

Iceberg: Formal definition

Definition. A *generic iceberg* is a sequence of disjoint measurable sets

$$\mathfrak{I} = \{B_{-h+1}, \ldots, B_{-1}, B_0, B_1, \ldots, B_{h-1}\}$$

such that $B_{j+1} \subseteq TB_j$ for $j \geq 0$ and $B_{j-1} \subseteq T^{-1}B_j$ for $j \leq 0$.

$\mathfrak{I}$ is composed of two icicles with common base B_0, one normal (*direct*) and one reverse, where *reverse icicle* is an icicle for T^{-1}.

We will use notation $\cup\mathfrak{I} = \bigcup_j B_j$.

Cyclic iceberg

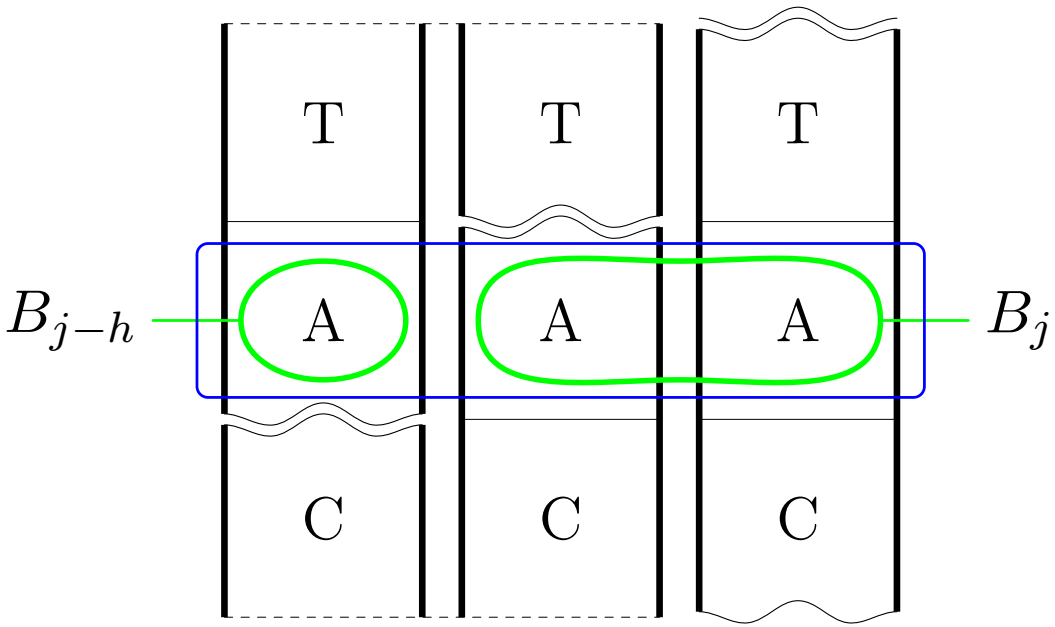


Figure: Partition $\bar{\mathfrak{I}} = \{B_j \cup B_{j-h}\}$ associated with a cyclic iceberg.

Cyclic iceberg

Definition. Let $\mathfrak{I}$ be a generic iceberg. We say that $\mathfrak{I}$ is *cyclic* if for any point $x \in B_0$ the total number of iterations towards future and past until leaving the iceberg is equal to h, i.e. $\#\{j \in \mathbb{Z}\colon\ T^j x \in B_j\} \equiv h$. Let us define the *cyclic iceberg partition*

$$\bar{\mathfrak{I}} = \{B_j \cup B_{j-h}\colon\ j = 0, 1, \ldots, h-1\}, \tag{1}$$

which is evidently refined by $\mathfrak{I}$.

Cyclic iceberg

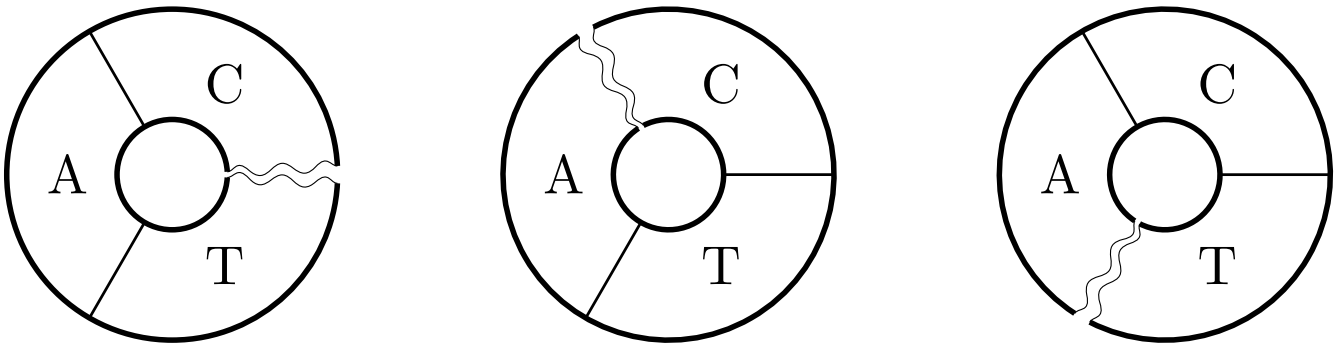


Figure: Unordered set of copies of the word "CAT" with different cut points.

Cyclic iceberg

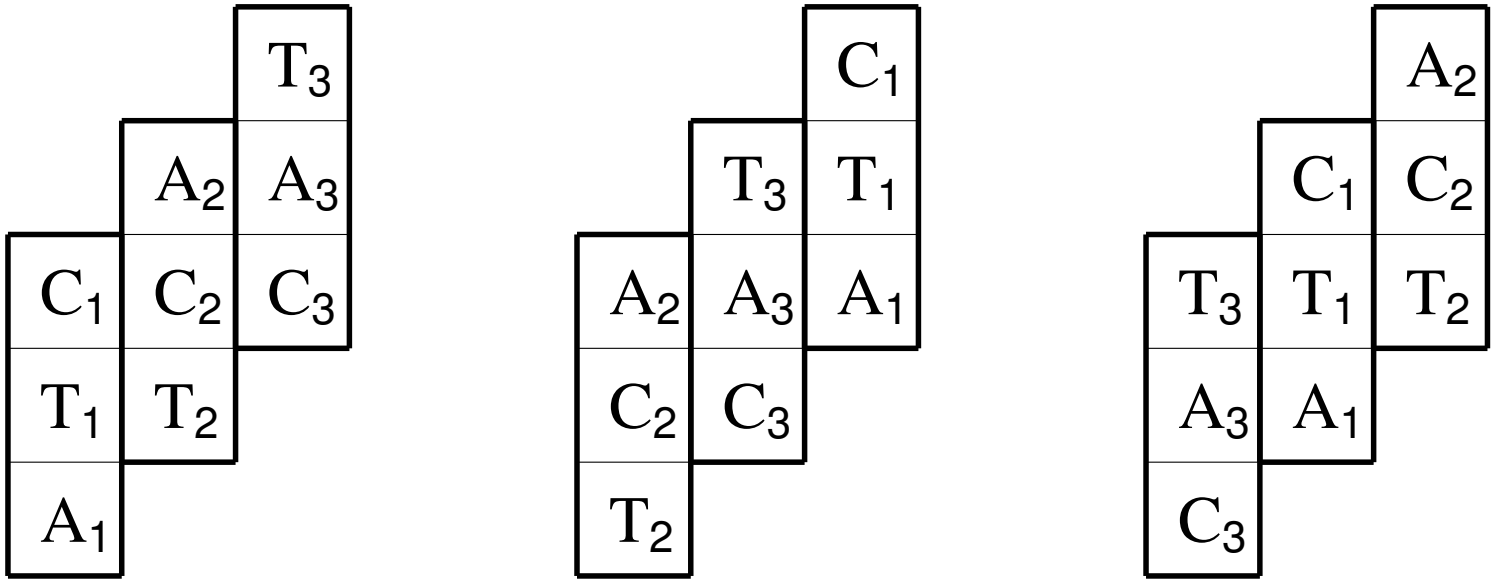
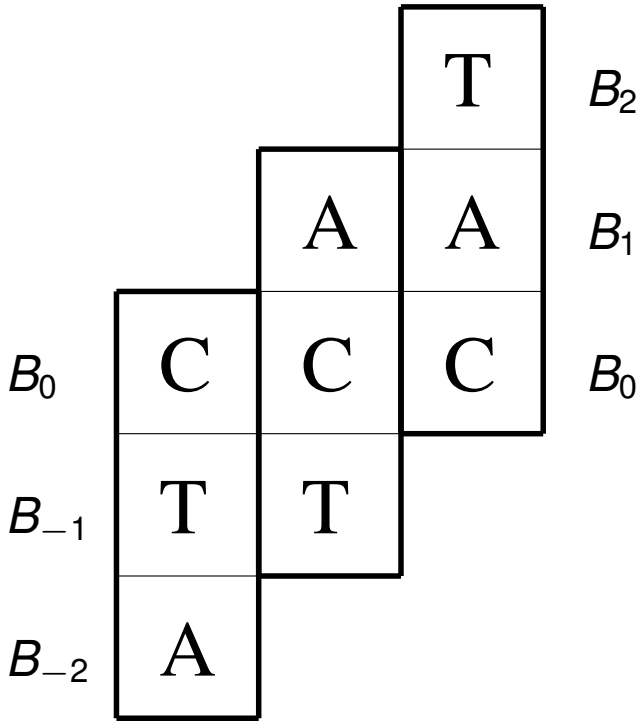


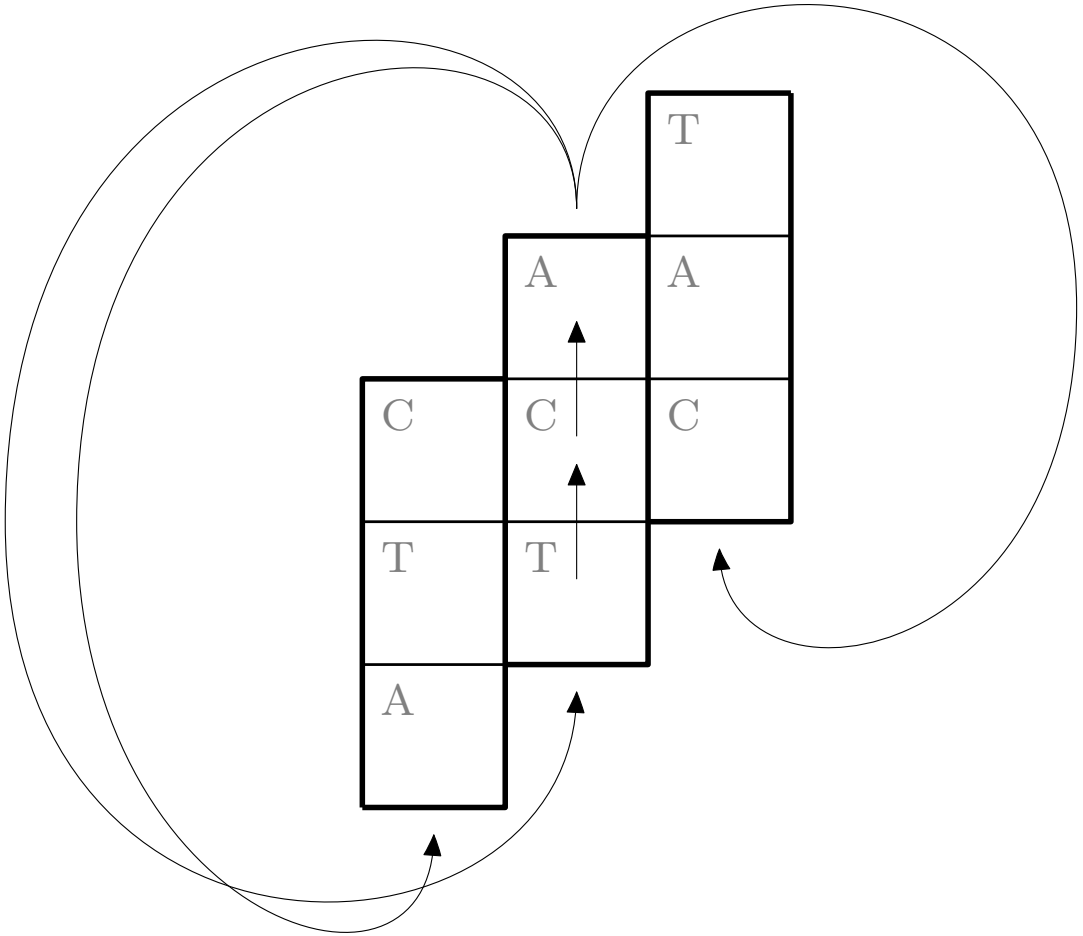
Figure: Cyclic rotation of a word leads to another choice of the base set and the letter defined to be the origin.

Interpretation of dynamics

Illustrating the behaviour of the transformation T we will gradually add more details to the picture



Interpretation of dynamics

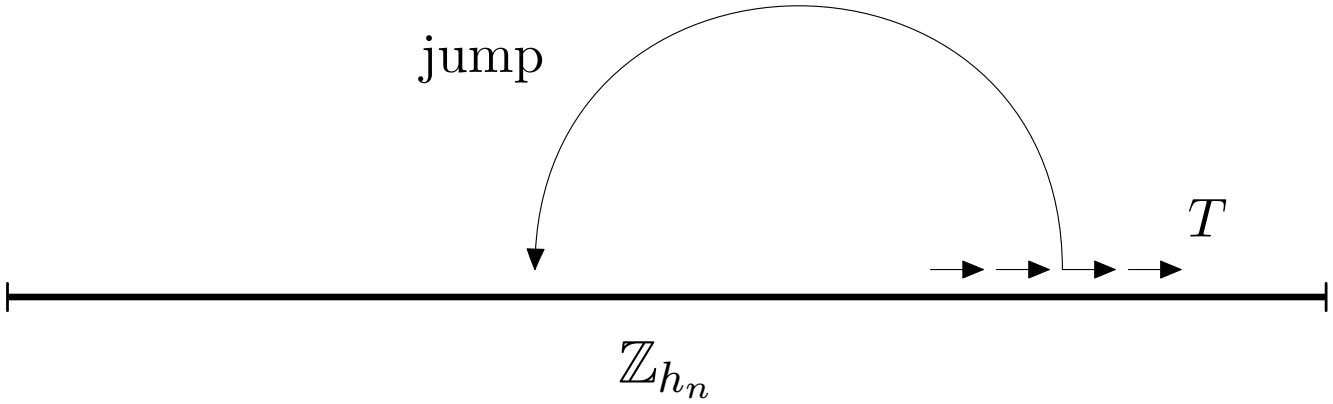


Interpretation of dynamics

Clearly the meaning of Poincarée map is to express the connection of two thin columns corresponding to a pair of adjacent rotated copies $\rho_{n,y}(W_n)\rho_{n,y+1}(W_n)$ which are subwords of W_{n+1}.

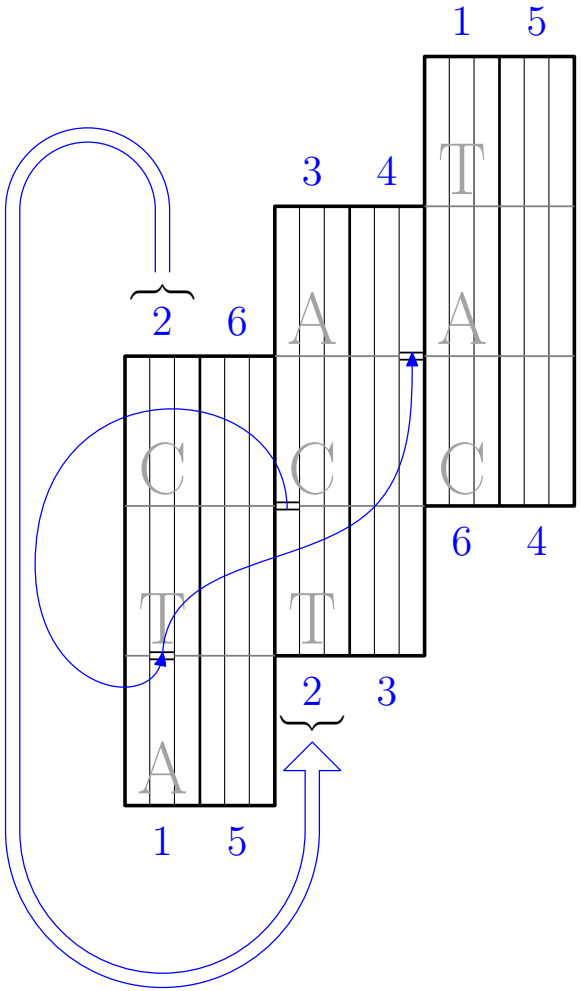
For example, if this pair is CAT.ATC the Poincarée map sends the top set in a thin column included in fat column "CAT" to the bottom set of some thin column in fat column "ATC".

Interpretation of dynamics



From the point of view of observer watching the coordinate x_n in the homogeneous space $\mathbb{Z}_{h_n}$ the following occurs: with probability $1-\varepsilon$ point x_n moves one step forward $x_n \mapsto x_n + 1$, and with small probability ε it *jumps* to any other point in $\mathbb{Z}_{h_n}$.

Interpretation of dynamics



Interpretation of dynamics

It is shown geometrically what happens when we rotate and concatenate copies of word

$$W_2 = \mathrm{CAT.ATC.TCA.TCA.CAT.ATC.}$$

and after create

$$W_3 = \mathrm{CATCACATATC\,|\,CATATCT\ .\ TCTCATCACATATC\,|\,CATA\ .\ ACATATC\,|\,CATATCTCATC}$$

Interpretation of dynamics

At the figure both first and second Poincaré maps are shown, and we can continue this procedure.

Let us mark the columns containing jumps as grayed.

Interpretation of dynamics

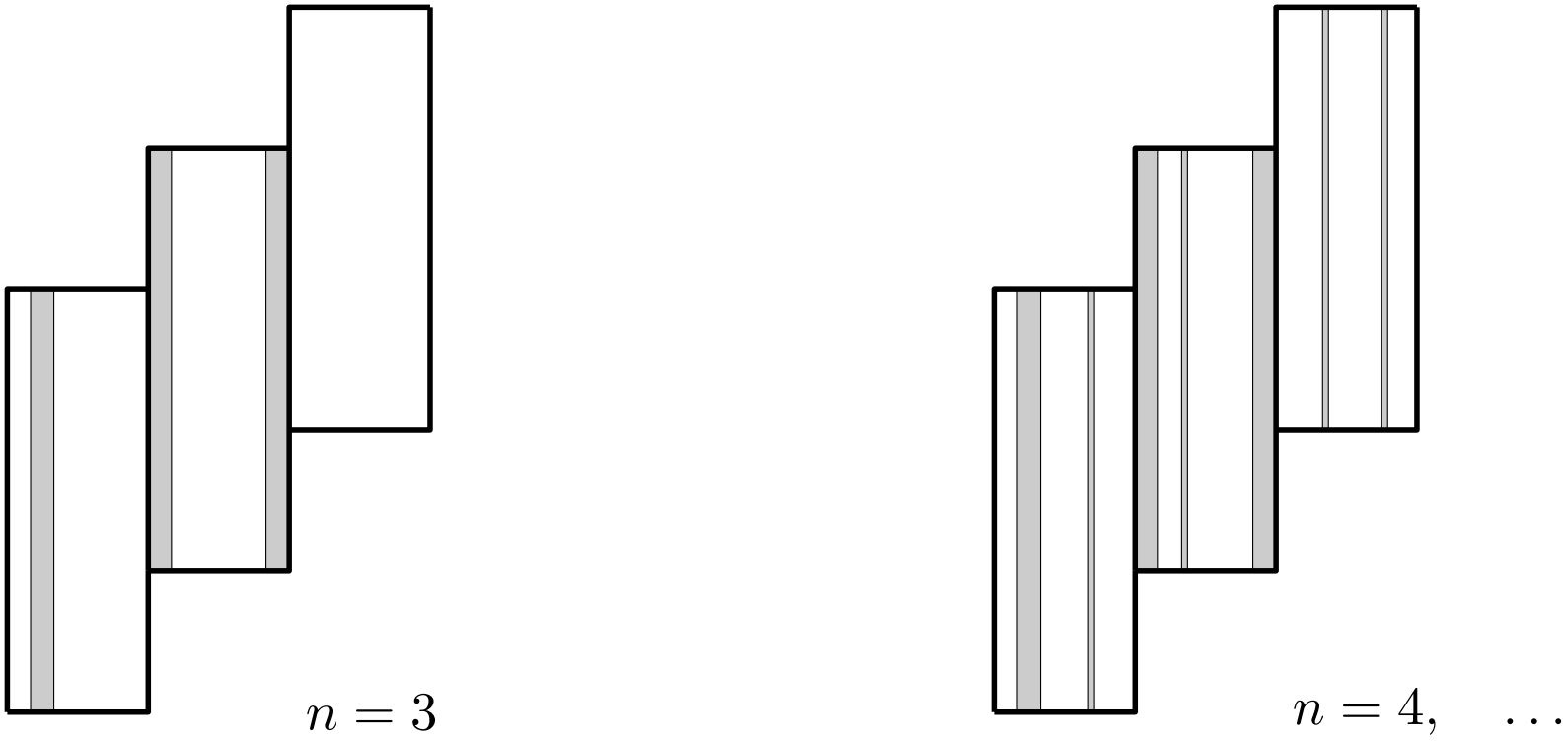


Figure: Grayed colomns contain jumps. If a point $x \in B_j$ and if x is located in the white area (*body*) then $Tx \in B_{j+1}$.

Iceberg approximation

We say that a measure preserving map T admits *iceberg approximation* if given a finite measurable partition associated with an alphabet $\mathbb{A}$ for any $\varepsilon > 0$ there exists a word W_ε in the alphabet $\mathbb{A}$ such that for $(1-\varepsilon)$-fraction of orbits (x_n) the subword of length $N(\varepsilon)$ in (x_n) starting from x_0 is ε-covered by rotations $\rho_\alpha(W_\varepsilon)$ of the word W_α.

Theorem. Iceberg approximation property is a dynamical invariant.

Iceberg approximation

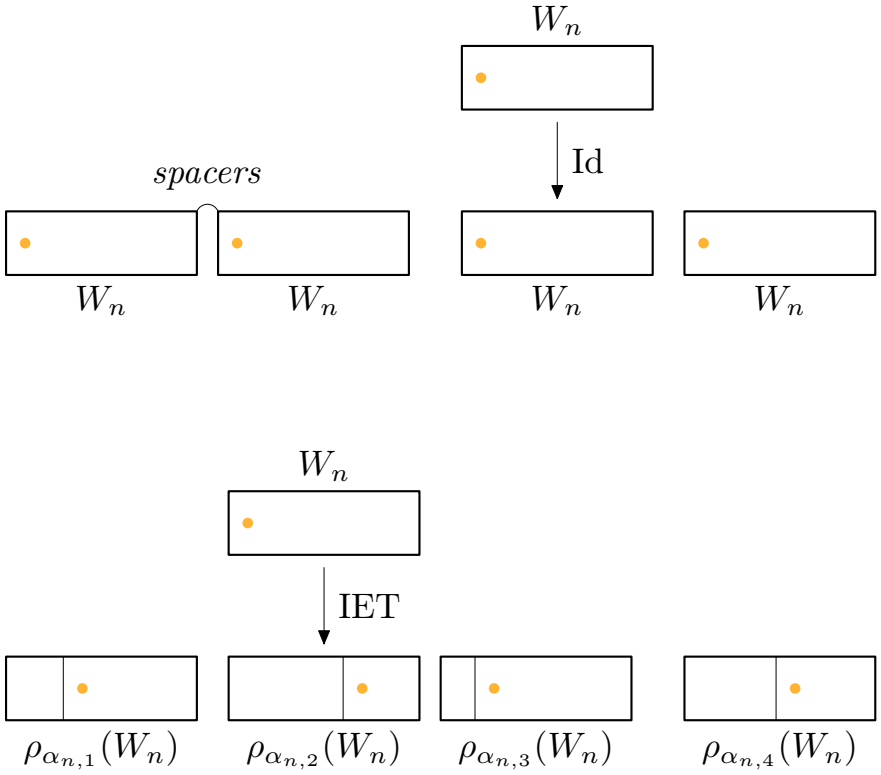


Figure: Comparing rank one and iceberg approximation

Iceberg approximation: Local rank 1/4

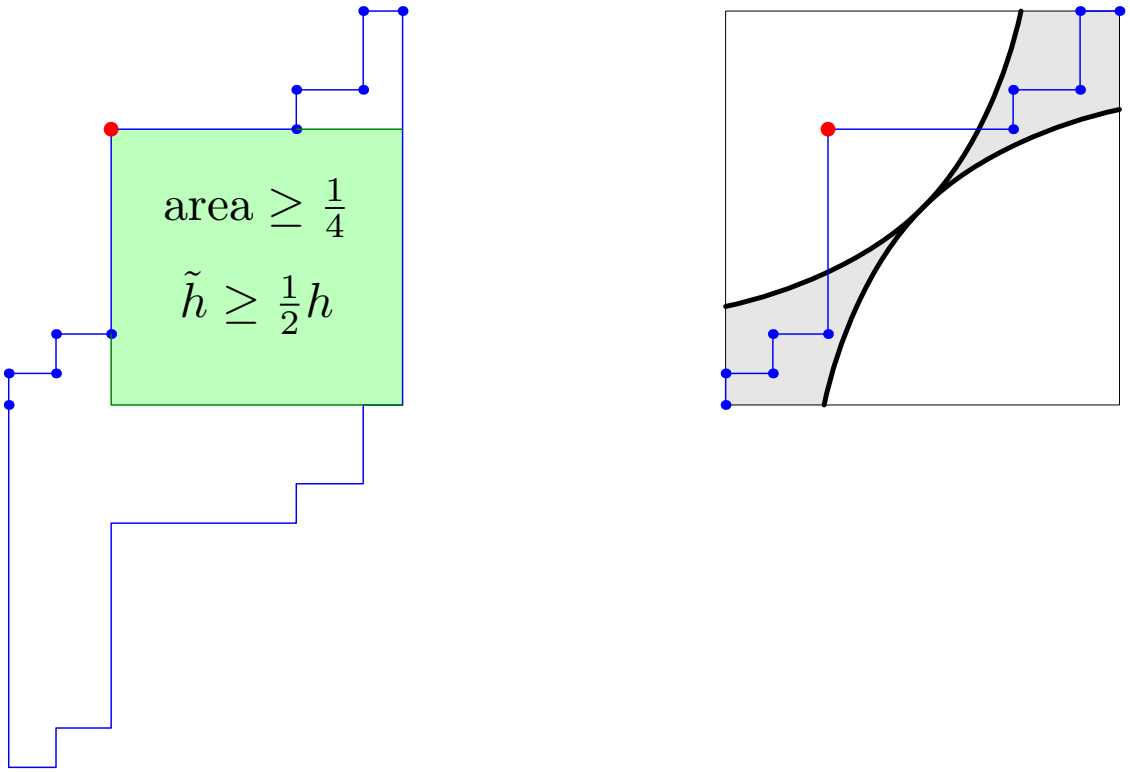


Figure: Proving 1/4-local rank property.

Iceberg approximation: Local rank 1/4

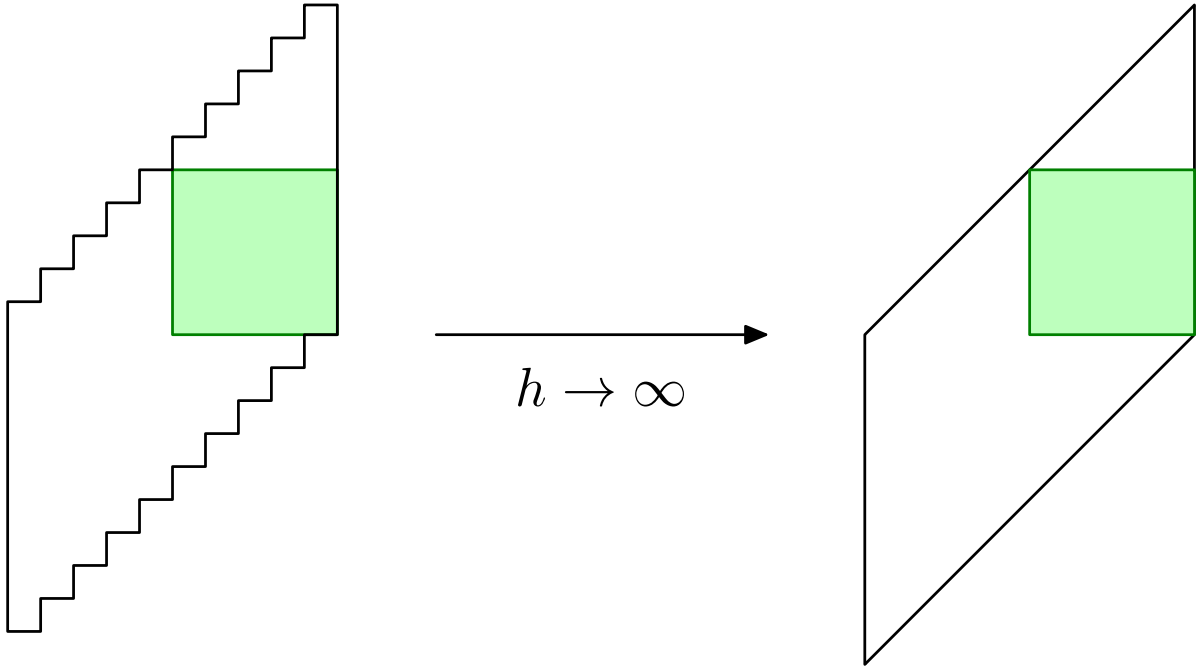


Figure: The area of maximal rectangle fit into the parallelogram is close to 1/4. Green rectangle corresponds to a Rokhlin subtower of the iceberg.

Simplicity of spectrum

Suppose that the maximal spectral multiplicity $m(T) \geq 2$. Then there exist functions f_1 and f_2 such that any cyclic subspace Z contains elements g_1, g_2 with the propertiy $\|g_1\| = \|g_2\| = a$, and satisfying

$$\|f_1 - g_1\|^2 + \|f_2 - g_2\|^2 \geq 2(1 + a^2 - 2a/\sqrt{2}).$$

We will show that for a class of transformations there exist a cyclic subspace approximating well both f_1 and f_2, and the contradiction will follow.

Simplicity of spectrum

Functions f_i can be approximatied by $\bar{\Im}_{n_0}$-measurable functions, and w.l.o.g. we can assume that f_1 are f_2 are $\bar{\Im}_{n_0}$-measurable, and $\bar{\Im}_n$-measurable for $n \geq n_0$.
Letting $b_n = \mathbf{1}_{B_{n,0}}$ we have

$$f = \sum_{j=0}^{h_n-1} f_{(n)}(j)\, S^j b_n = \sum_{j=-(h_n-1)/2}^{(h_n-1)/2} f_{(n)}(j)\, S^j b_n, \qquad n \geq n_0,$$

where by definition $SB_j = B_{j+1}$ is the operator in $L^2(X_n)$ corresponding to the rotation $t \mapsto t+1$.

Simplicity of spectrum

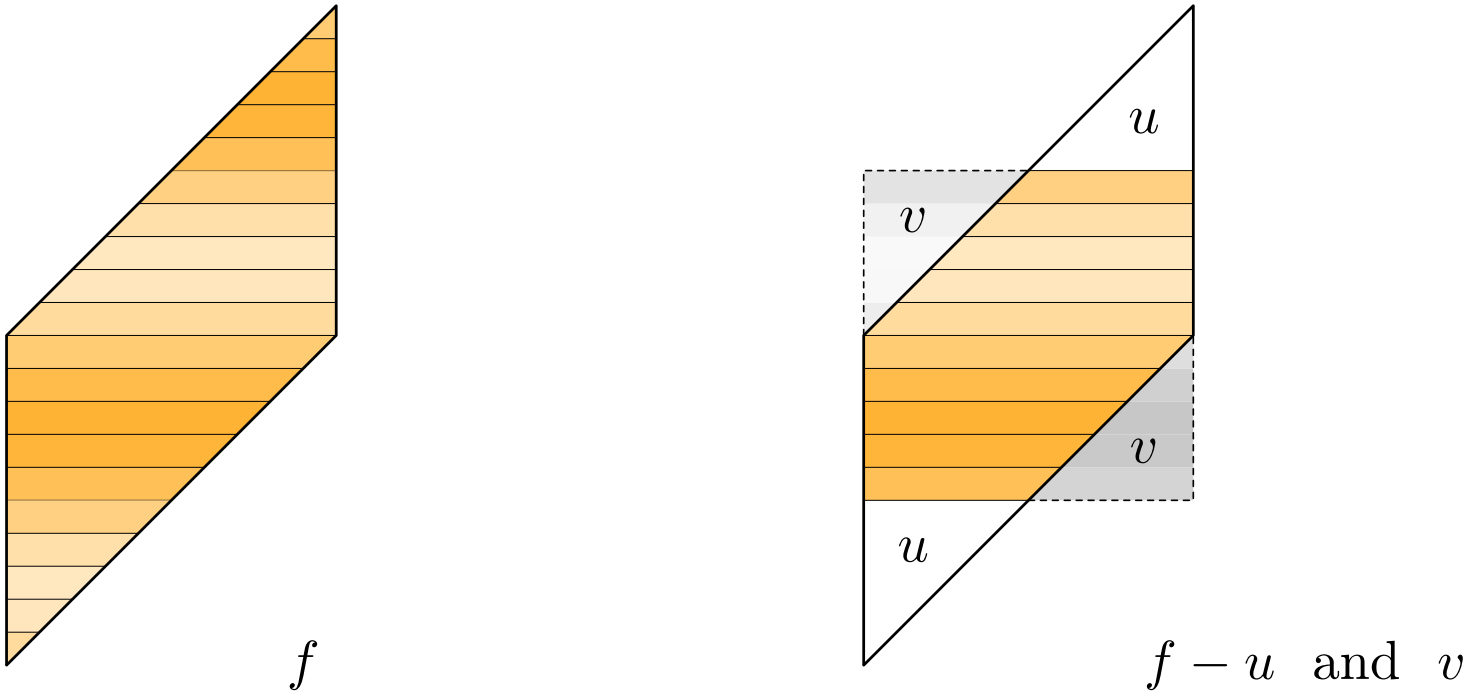


Figure: Proving simplicity of spectrum

Simplicity of spectrum

Idea. The following way of proving simplity of spectrum one can call "3/4-strategy". We approximate function f by the iterations $T^j b_n$ of the indicatior of the base level $B_{n,0}$, where $j = -(h_n - 1)/2, \ldots, (h_n - 1)/2$, on the subset of the phase space X of measure $\approx 3/4$.

Remark that the estimates below used by this approach is not a priory necessary, but sufficient for simplicity of spectrum.

Simplicity of spectrum

We can assume that $\int f\,d\mu = 0$. Consider sets G_n and E_n of measure $\approx 3/4$ and $\approx 1/4$ respectively, where G_n corresponds to the central area and E_n is the union of two triangular areas remote from the base level,

$$\cup\mathfrak{I}_n = G_n \cup E_n, \qquad G_n = \bigcup_{j=-(h_n-1)/2}^{(h_n-1)/2} B_{n,j},$$

$$E_n = \bigcup_{j=-h_n+1}^{-(h_n-1)/2-1} B_{n,j} \quad \cup \quad \bigcup_{j=(h_n-1)/2+1}^{h_n-1} B_{n,j}$$

Simplicity of spectrum

The function f is approximated by the function

$$g = \sum_{j=-(h_n-1)/2}^{(h_n-1)/2} f_{(n)}(j)\, T^j b_n$$

and g can be represented in the following way:

$$g = f - u + v, \qquad f - u = f|_{G_n}, \quad u = f_{E_n},$$

and v is uniquely definied from the equation. The meaning of the function v can be explained as follows.

Simplicity of spectrum

Walking from the base of the iceberg under action of T a point x is moving in the vertical direction, and the value $f(T^j x)$ is recovered by the index of level, namely, $f(T^j x) = f|_{B_j}$,

$$T^j b_n = \mathbf{1}_{B_{n,j}} + \xi_j, \qquad v = \sum_{j=-(h_n-1)/2}^{(h_n-1)/2} f_{(n)}(j)\, \xi_j,$$

and

$$u = \sum_{j=-h_n+1}^{-(h_n-1)/2-1} f_{(n)}(j)\, \mathbf{1}_{B_j} + \sum_{j=(h_n-1)/2+1}^{h_n-1} f_{(n)}(j)\, \mathbf{1}_{B_j}.$$

Simplicity of spectrum

Lemma. Suppose that $\langle u, v\rangle \to 0$ and $\langle f, v\rangle \to 0$ as $n \to \infty$ then asymptotically $\|f - g\|^2 \to 1/2$ and $\|g\| = (1 + o(1))\|f\|$. In particular, T has simple spectrum.

Proof. Assume that f has zero mean and $\|f\| = 1$. Let us notice that $v(x) = u(\Phi x)$ for some measure preserving invertible map Φ, hence, $\|v\|^2 = \|u\|^2$.

Simplicity of spectrum

We have

$$a^2 = \|g\|^2 = \|f - u + v\|^2 =$$
$$= \|f\|^2 + \|u\|^2 + \|v\|^2 - 2\,\mathrm{Re}\,\langle f,\, u\rangle + 2\,\mathrm{Re}\,\langle f,\, v\rangle - 2\,\mathrm{Re}\,\langle u,\, v\rangle \approx$$
$$\approx \|f\|^2 + 2\|u\|^2 - 2\,\mathrm{Re}\,\langle f,\, u\rangle = \|f\|^2 = 1,$$

and

$$\|u\|^2 \approx \frac{1}{4}\|f\|^2, \qquad \langle f,\, u\rangle = \|u\|^2.$$

Simplicity of spectrum

Now let us estimate $\|f - g\|$:

$$\|f - g\|^2 = \| - u + v\|^2 \approx \|u\|^2 + \|v\|^2 \approx \frac{1}{2}\|f\|^2 = \frac{1}{2}.$$

To establish the second statement of the lemma let us take into accont that we use the same estimates both for f_1 and f_2. Thus, we have to analyze the following inequality:

$$\|f - g\|^2 \geq 1 + a^2 - a\sqrt{2}.$$

Simplicity of spectrum

Using results of the above calculations we have

$$\|f-g\|^2 \geq 1 + a^2 - a\sqrt{2},$$
$$\frac{1}{2} \geq 2 - \sqrt{2},$$
$$\sqrt{2} \geq \frac{3}{2},$$

and we come to contradiction.

Simplicity of spectrum: Random iceberg map

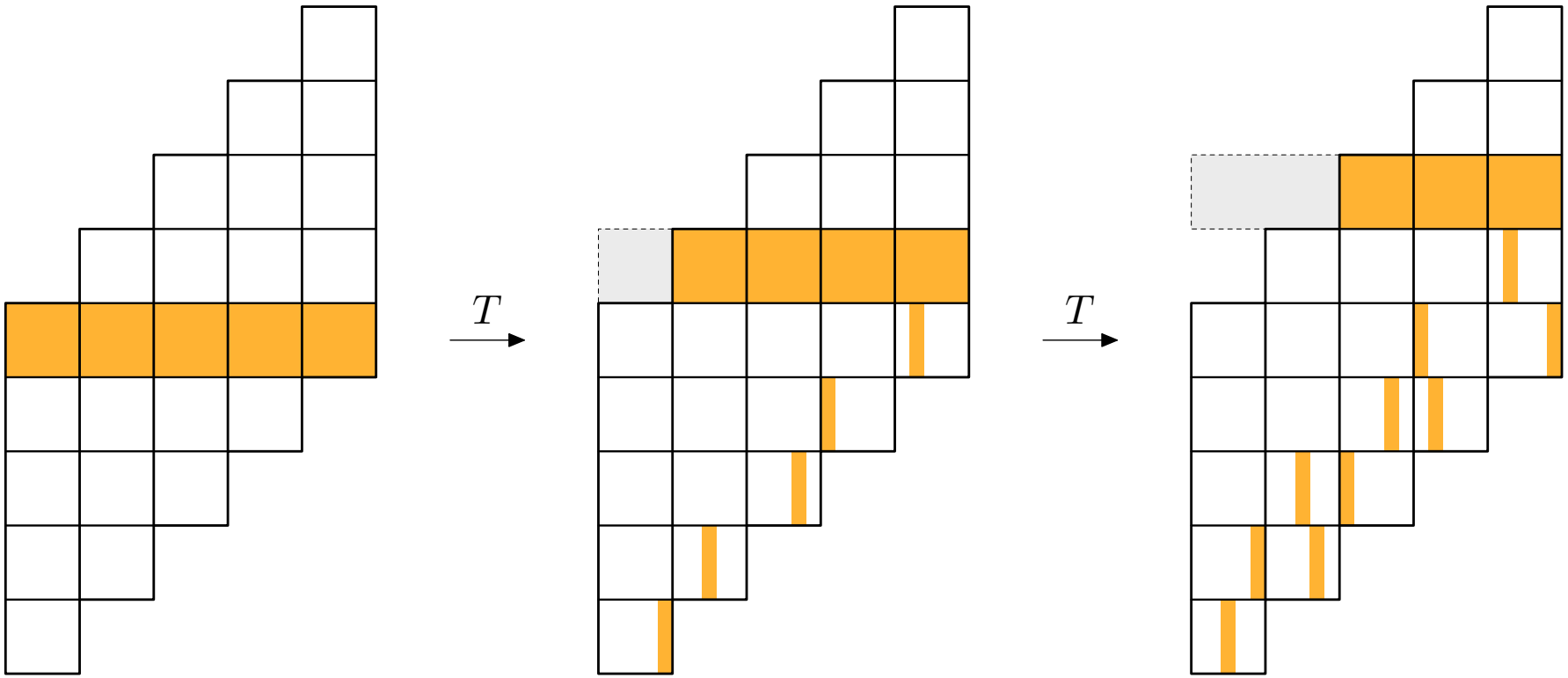


Figure: Iterates of the base level of an iceberg.

Simplicity of spectrum: Random iceberg map

Lemma. Suppose that

$$\left\| \mu(TV_{n,k,k-1} \mid V_{n,s,s-h}) - \frac{1}{h_n} \right\|_1 = o\left(\frac{1}{h_n}\right). \tag{2}$$

Then $\langle u, v \rangle \to 0$ and $\langle f, v \rangle \to 0$ as $n \to \infty$.

This lemma immediately implies that T has simple spectrum.

Thank you!